

Chapter **3**

Calibration of Stochastic Volatility
Models Using Statistical Tests

Section 3.1

The Use of Statistical Tests to Calibrate the Black-Scholes Model

[Description] *A new method to solve the calibration problem for the Black-Scholes asset price dynamics model is proposed. The data used in the calibration problem are the observations of the asset price on a finite set of equispaced (known) discrete time values. Statistical tests are used to obtain estimates with statistical significance of the two parameters of the Black-Scholes model, that is of the volatility and of the drift. The consequences of these estimates on the option pricing problem are investigated. In particular the pricing problem for options with uncertain volatility in the Black-Scholes framework is revisited and a statistical significance is associated to the option price intervals determined using the Black-Scholes-Barenblatt equations. Numerical experiments with synthetic and real data are presented. The real data considered are the daily closing values of the S&P500 index and of the associated European call and put option prices in the year 2005. The method proposed to calibrate the Black-Scholes dynamics model can be extended to other stochastic dynamical systems used as models in science and engineering.*

[Paper] *Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2012). The use of statistical tests to calibrate the Black-Scholes asset dynamics model applied to pricing options with uncertain volatility, Journal of Probability and Statistics, Volume 2012, Article ID 931609, 20 pages.*

[Website] <http://www.econ.univpm.it/recchioni/finance/w11>

3.1.1 Outline of the Presentation

- We consider the calibration problem for the Black-Scholes asset price dynamics model (Black et al. 1973).
- The data used in the calibration problem are the observations of the asset price on a finite set of equispaced (known) discrete time values.
- The method proposed to solve the calibration problem uses statistical tests in order to obtain estimates with statistical significance of the two parameters of the Black-Scholes model, i.e.: the volatility and the drift.
- We present the consequences of these estimates (with significance levels) on the option pricing problem. In particular we revisit the pricing problem for options with uncertain volatility in the Black-Scholes framework associating a statistical significance to the option price intervals determined using the Black-Scholes-Barenblatt (BSB) equation.
- Numerical experiments with real data are presented.

3.1.2 The Calibration Problem

We want to estimate (with statistical significance) the volatility and the drift parameters of the Black-Scholes asset price dynamics model starting from a set of data.

We use as set of data the observations of the asset price on a finite set of (known) equispaced discrete time values.

The solution of the calibration problem proposed uses the Student's T and the χ^2 statistical tests, in order to provide values of the volatility and

of the drift parameters of the Black-Scholes model with a *statistical significance* associated.

In a statistical test the statistical significance α , $0 < \alpha < 1$, is the maximum probability of rejecting the (null) hypothesis of the test when the hypothesis is true.

We consider the effects of these significance levels on the pricing problem for options with uncertain volatility. We assume that the (uncertain) volatility belongs to a known interval and we determine the corresponding price intervals for the (European vanilla) option prices using the Black-Scholes-Barenblatt (BSB) equation (see, for example, Avellaneda et al. 1995, Lyons, 1995).

Thank to our methodology statistical significance levels can be attributed to the option price intervals determined using the BSB equation.

Using these tools we study the data time series made of the daily closing values of the *S&P500* index and of the associated European vanilla call and put option prices in the year 2005.

Remarks

1. The calibration problem for the Black-Scholes asset dynamics model is an inverse problem for a stochastic dynamical system defined by a stochastic differential equation.
2. The use of statistical tests in the solution of the calibration problem for stochastic dynamical systems is an interesting way of approaching these inverse problems. It can be used in many application contexts different from mathematical finance. The Black-Scholes model is very simple and can be studied with elementary statistical tests (Student's T, χ^2). The study of more general models with this methodology requires the

development of ad hoc statistical tests.

3.1.3 The Calibration Problem for the Black-Scholes Model

Let $S_t > 0$ denote the asset price at time $t \geq 0$. The Black-Scholes model assumes that $S_t, t > 0$, satisfies:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, t > 0,$$

$$S_0 = \hat{S}_0,$$

where μ, σ are real parameters, μ is the drift, $\sigma > 0$ is the volatility, $W_t, t > 0$, is the standard Wiener process, $W_0 = 0, dW_t, t > 0$, is its stochastic differential and $\hat{S}_0 > 0$ is a given random variable. We assume \hat{S}_0 concentrated in a point with probability one.

The real parameters μ, σ are the unknowns of the calibration problem.

Let $G_t = \ln \left(\frac{S_t}{\hat{S}_0} \right), t > 0$, be the log-return at time t of the asset whose price is $S_t, t > 0$. The process $G_t, t > 0$, satisfies:

$$dG_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t, t > 0,$$

- $G_t = \ln \left(\frac{S_t}{\hat{S}_0} \right), t > 0$, is a generalized Wiener process with constant drift $\mu - \frac{\sigma^2}{2}$ and constant volatility $\sigma > 0$;
- for $t \geq 0, \tau > 0$, the increment in $G_t = \ln \left(\frac{S_t}{\hat{S}_0} \right)$ occurring between time t and time $t + \tau$, is a Gaussian random variable with mean $\left(\mu - \frac{\sigma^2}{2} \right) \tau$ and variance $\sigma^2 \tau$, i.e.:

$$G_{t+\tau} - G_t = \ln S_{t+\tau} - \ln S_t \sim \mathcal{N} \left(\left(\mu - \frac{\sigma^2}{2} \right) \tau, \sigma^2 \tau \right), t \geq 0, \tau > 0,$$

where, for M, V real constants, $\mathcal{N}(M, V^2)$ denotes the Gaussian distribution with mean M and variance V^2 .

Let $\Delta t > 0$ be a time increment and $t_i = i\Delta t, i = 0, 1, \dots, n$ be a discrete set of equispaced time values. We define X_{t_i} , the asset price log-return increment when t goes from t_{i-1} to $t_i, i = 1, 2, \dots, n$, as follows:

$$X_{t_i} = \ln \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right), \quad i = 1, 2, \dots, n.$$

The random variables $X_{t_i}, i = 1, 2, \dots, n$, are independent identically distributed (i.i.d.) Gaussian random variables with mean M and variance V^2 where:

$$M = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t, \quad V^2 = \sigma^2 \Delta t.$$

That is we have:

$$X_{t_i} \sim \mathcal{N} \left(\left(\mu - \frac{\sigma^2}{2} \right) \Delta t, \sigma^2 \Delta t \right), \quad i = 1, 2, \dots, n.$$

3.1.4 The Calibration Problem

Given $\Delta t > 0$, a statistical significance level α , $0 < \alpha < 1$, and the asset price \hat{S}_i observed at time $t = t_i = i\Delta t$, $i = 0, 1, \dots, n$, determine two intervals where the parameters of the Black-Scholes model μ and $\sigma > 0$ belong with the given significance level α .

The observed log-return increments $\hat{x}_i = \ln\left(\frac{\hat{S}_i}{\hat{S}_{i-1}}\right)$, $i = 1, 2, \dots, n$, are a sample of n observations taken respectively from X_{t_i} , $i = 1, 2, \dots, n$, that is taken from a set of i.i.d. Gaussian random variables. Using this data sample, through the Student's T test and the χ^2 test respectively, we determine two intervals where the mean M and the variance V^2 of these random variables belong with the given significance level α .

From the knowledge of the intervals determined for M and V^2 , the corresponding intervals for μ and σ can be easily recovered.

Remark

In many circumstances it is more practical to try to determine an interval of variability for the drift μ and for the volatility σ of the Black-Scholes model instead than trying to determine their "exact" values.

3.1.5 The Student's T Test and the χ^2 Test for the Gaussian Random Variable

In the most simple circumstances, like the one considered here, inferences about the mean M of a Gaussian distribution $\mathcal{N}(M, V^2)$ are based on the Student's T test and inferences about the variance V^2 of a normal distribution $\mathcal{N}(M, V^2)$ are based on the χ^2 test.

Let $n \geq 2$ and let Y_1, Y_2, \dots, Y_n be a set of i.i.d. random variables whose distribution is $\mathcal{N}(M, V^2)$.

Let

$$\bar{Y} = \frac{Y_1 + Y_2 + \dots + Y_n}{n} = \frac{1}{n} \sum_{i=1}^n Y_i,$$

$$\Sigma^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

3.1.6 The Student's T Test

Given two real numbers M_1, M_2 , such that $M_1 < M_2$, we are interested in testing the following (composite) hypotheses:

$$H_0 : M_1 \leq M \leq M_2,$$

versus

$$H_1 : M < M_1 \text{ or } M > M_2.$$

The corresponding decision table is:

1. accept $H_0 : M_1 \leq M \leq M_2$;
2. reject $H_0 : M_1 \leq M \leq M_2$.

The two test statistics are:

$$T_1 = \frac{\bar{Y} - M_1}{\Sigma/\sqrt{n}}, \quad T_2 = \frac{\bar{Y} - M_2}{\Sigma/\sqrt{n}}.$$

Given a significance level α , $0 < \alpha < 1$, the decision rules are:

1. accept $H_0 : M_1 \leq M \leq M_2$, with significance level α , if on the data sample we have: $T_1 > -t_{n-1,\alpha/2}$ and $T_2 < t_{n-1,\alpha/2}$;
2. reject $H_0 : M_1 \leq M \leq M_2$, with significance level α , if on the data sample we have: $T_1 \leq -t_{n-1,\alpha/2}$ or $T_2 \geq t_{n-1,\alpha/2}$.

The number $t_{n-1,\alpha/2}$ is the solution of:

$$P(-t_{n-1,\alpha/2} \leq T \leq t_{n-1,\alpha/2}) = 1 - \alpha,$$

where $P(\cdot)$ denotes the probability of \cdot and T is a random variable with Student's T distribution with $n - 1$ degrees of freedom.

3.1.7 The χ^2 Test

Given two positive numbers V_1, V_2 , such that $V_1^2 < V_2^2$, let us consider the test of the following (composite) hypotheses:

$$H_0 : V_1^2 \leq V^2 \leq V_2^2,$$

versus

$$H_1 : V^2 < V_1^2 \text{ or } V^2 > V_2^2.$$

The corresponding decision table is:

1. accept $H_0 : V_1^2 \leq V^2 \leq V_2^2$;
2. reject $H_0 : V_1^2 \leq V^2 \leq V_2^2$.

The two test statistics are:

$$\chi_1^2 = \frac{(n-1)\Sigma^2}{V_1^2}, \quad \chi_2^2 = \frac{(n-1)\Sigma^2}{V_2^2}.$$

Given the significance level α , $0 < \alpha < 1$, the decision rules are:

1. accept $H_0 : V_1^2 \leq V^2 \leq V_2^2$, with significance level α , if on the data sample we have: $\chi_1^2 > \chi_{n-1, \alpha/2}$ and $\chi_2^2 < \gamma_{n-1, \alpha/2}$;
2. reject $H_0 : V_1^2 \leq V^2 \leq V_2^2$, with significance level α , if on the data sample we have: $\chi_1^2 \leq \chi_{n-1, \alpha/2}$ or $\chi_2^2 \geq \gamma_{n-1, \alpha/2}$,

where:

$$P(\chi_{n-1, \alpha/2} \leq \chi^2) = 1 - \frac{\alpha}{2},$$

$$P(\chi^2 \leq \gamma_{n-1, \alpha/2}) = 1 - \frac{\alpha}{2},$$

where χ^2 is a random variable with χ^2 distribution with $n - 1$ degrees of freedom.

Given a significance level α , $0 < \alpha < 1$, we can perform statistical tests on the variance V^2 and on the mean M of the random variables $X_{t_i} = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$, $i = 1, 2, \dots, n$, starting from the data sample $\hat{x}_i = \ln\left(\frac{\hat{S}_i}{\hat{S}_{i-1}}\right)$, $i = 1, 2, \dots, n$, using the χ^2 test and the Student's T test respectively.

This implies that given α , $0 < \alpha < 1$, we can accept or reject, with significance level α , the hypotheses:

$$\sigma_1 \leq \sigma \leq \sigma_2,$$

and

$$\mu_1 \leq \mu \leq \mu_2,$$

where

$$\sigma_i = \frac{V_i}{\sqrt{\Delta t}}, \quad \text{and} \quad \mu_i = \frac{M_i}{\Delta t} + \frac{V_i^2}{2\Delta t}, \quad i = 1, 2,$$

simply translating to σ and μ the results on V^2 and M obtained with the statistical tests described previously (see Fatone et al. 2012 for further details).

3.1.8 Large Option Prices with Uncertain Volatility and Statistical Significance

Given a significance level α , $0 < \alpha < 1$, assuming that the hypothesis $H_0 : \sigma_1 \leq \sigma \leq \sigma_2$ is accepted with significance level α , determine (with significance level α) the range where the value of an European vanilla option lies.

The answer to this question follows from the work of Avellaneda, Levy and Parás 1995 and of Lyons 1995. These authors propose a way to price options in the Black-Scholes model when the volatility σ is not known exactly, but it is known that:

$$\sigma_1 \leq \sigma \leq \sigma_2.$$

In Avellaneda et al. 1995 and Lyons, 1995 significance levels are not considered.

We limit our attention to European vanilla *call* and *put* options.

Let t be the time variable, S be the asset price, $\mathcal{T} > 0$ be the expiration date of the option to be priced and r be the risk free interest-rate. Moreover let g be the pay-off function of the option. For example, if K denotes the strike price of the option, for an European vanilla call option we have

$g(S) = \max(S - K, 0)$, $S > 0$, while for an European vanilla put option we have $g(S) = \max(K - S, 0)$, $S > 0$.

In Avellaneda et al. 1995 and Lyons, 1995 it is shown that in the Black-Scholes framework when $\sigma_1 \leq \sigma \leq \sigma_2$ there exists an interval $[\mathcal{V}_1, \mathcal{V}_2]$ depending on S and t , such that the price $\mathcal{V} = \mathcal{V}(S, t)$, $S > 0$, $0 < t \leq \mathcal{T}$, of the option lies in this interval, that is:

$$\mathcal{V}_1(S, t) \leq \mathcal{V}(S, t) \leq \mathcal{V}_2(S, t), \quad S > 0, \quad 0 < t \leq \mathcal{T}.$$

The worst-case option value $\mathcal{V}_1(S, t)$, $S > 0$, $0 < t \leq \mathcal{T}$, satisfies the following nonlinear partial differential equation known as Black-Scholes-Barenblatt (BSB) equation:

$$\frac{\partial \mathcal{V}_1}{\partial t} + \frac{1}{2} a(\Gamma_1)^2 S^2 \frac{\partial^2 \mathcal{V}_1}{\partial S^2} + rS \frac{\partial \mathcal{V}_1}{\partial S} - r\mathcal{V}_1 = 0, \quad S > 0, \quad 0 < t < \mathcal{T},$$

with final condition:

$$\mathcal{V}_1(S, \mathcal{T}) = g(S), \quad S > 0,$$

where

$$\Gamma_1 = \frac{\partial^2 \mathcal{V}_1}{\partial S^2},$$

and

$$a(\Gamma_1) = \begin{cases} \sigma_2, & \text{if } \Gamma_1 \leq 0, \\ \sigma_1, & \text{if } \Gamma_1 > 0. \end{cases}$$

Similarly, the best-case option value $\mathcal{V}_2(S, t)$, $S > 0$, $0 < t \leq \mathcal{T}$, satisfies the following BSB equation:

$$\frac{\partial \mathcal{V}_2}{\partial t} + \frac{1}{2} b(\Gamma_2)^2 S^2 \frac{\partial^2 \mathcal{V}_2}{\partial S^2} + rS \frac{\partial \mathcal{V}_2}{\partial S} - r\mathcal{V}_2 = 0, S > 0, 0 < t < \mathcal{T},$$

with final condition:

$$\mathcal{V}_2(S, \mathcal{T}) = g(S), \quad S > 0,$$

where

$$\Gamma_2 = \frac{\partial^2 \mathcal{V}_2}{\partial S^2},$$

and

$$b(\Gamma_2) = \begin{cases} \sigma_2, & \text{if } \Gamma_2 \geq 0, \\ \sigma_1, & \text{if } \Gamma_2 < 0. \end{cases}$$

Remarks

1. The BSB equation reduces to the Black-Scholes equation when $\sigma_1 = \sigma_2$.
2. For a general pay-off function these equations don't have a closed-form solution and must be solved numerically.
3. When a *call* or a *put* option is considered, due to the convexity of the corresponding pay-off functions $g(S)$, $S > 0$, and to the properties of the parabolic equations (such as the BSB equation), it can be shown that the functions $\frac{\partial^2 \mathcal{V}_1}{\partial S^2}$ and $\frac{\partial^2 \mathcal{V}_2}{\partial S^2}$ do not change sign for $S > 0, 0 < t < \mathcal{T}$. That is when $S > 0, 0 < t < \mathcal{T}$, the functions $\frac{\partial^2 \mathcal{V}_1}{\partial S^2}$ and $\frac{\partial^2 \mathcal{V}_2}{\partial S^2}$ keep the sign that they have at $t = \mathcal{T}$, therefore, when a *call* or a *put* option

is considered, the BSB equations reduce to the Black-Scholes equation.

The Black-Scholes equation is linear and, when simple final conditions are imposed, can be solved explicitly (see Fatone et al. 2012).

For example if we consider an European *call* option the worst-case option value \mathcal{V}_1 is the solution of the following problem:

$$\frac{\partial \mathcal{V}_1}{\partial t} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 \mathcal{V}_1}{\partial S^2} + rS \frac{\partial \mathcal{V}_1}{\partial S} - r\mathcal{V}_1 = 0, \quad S > 0, 0 < t < \mathcal{T},$$

$$\mathcal{V}_1(S, \mathcal{T}) = \max(S - K, 0), \quad S > 0,$$

and similarly the best-case call option value \mathcal{V}_2 satisfies:

$$\frac{\partial \mathcal{V}_2}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 \mathcal{V}_2}{\partial S^2} + rS \frac{\partial \mathcal{V}_2}{\partial S} - r\mathcal{V}_2 = 0, \quad S > 0, 0 < t < \mathcal{T},$$

$$\mathcal{V}_2(S, \mathcal{T}) = \max(S - K, 0), \quad S > 0,$$

and, as it is well known, these problems have *explicit solutions* given by the Black-Scholes formula.

First of all we assume that a “true” value of the volatility σ exists even if it is unknown. From the fact that in the Black-Scholes model *the price* \mathcal{V} *of an option is a monotonically increasing function of the volatility* σ we can conclude that when

$$\sigma_1 \leq \sigma \leq \sigma_2, \quad \text{with significance level } \alpha.$$

we have:

$$“\mathcal{V}_1(S, t) \leq \mathcal{V}(S, t) \leq \mathcal{V}_2(S, t), \quad S > 0, 0 < t < \mathcal{T}”$$

with significance level α ,

where $\mathcal{V}_1(S, t)$, $\mathcal{V}_2(S, t)$, $S > 0$, $0 < t < \mathcal{T}$, are the solutions of the appropriate BSB equations.

3.1.9 Some Numerical Results on Real Data

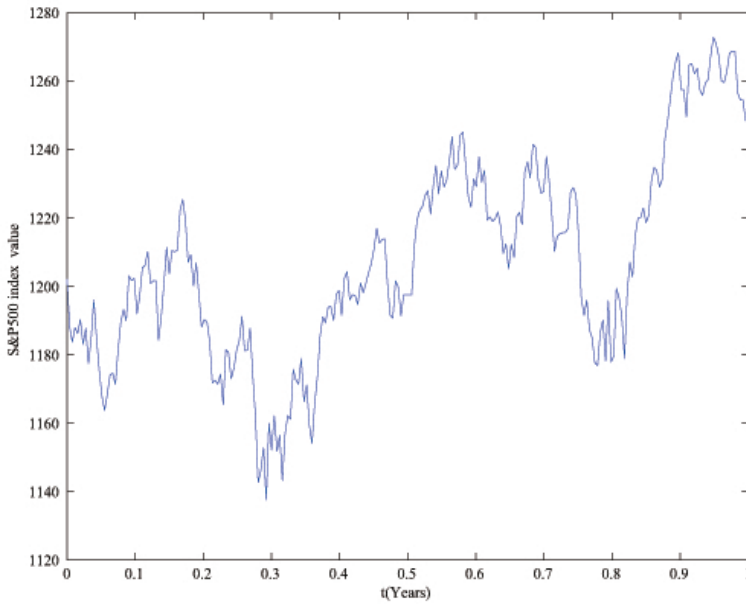
1. Study of the variance and of the drift of the *S&P500* index during the year 2005

The real data studied are the 2005 daily data of the U.S. *S&P500* index and of the prices of European vanilla call and put options on this index. We remind that the U.S. *S&P500* index is one of the leading indices of the New York Stock Exchange.

More specifically we consider the daily closing values of the *S&P500* index and of the bid prices of the vanilla European call and put options on the *S&P500* index during the period of about 12 months going from January 3, 2005 to December 30, 2005. In this period we have more than 153.000 option prices. We limit our study to the call and the put prices corresponding to options that have a positive volume (i.e. a positive number of contracts) traded the day corresponding to the price considered and a positive bid price. These prices are 46.823 options prices.

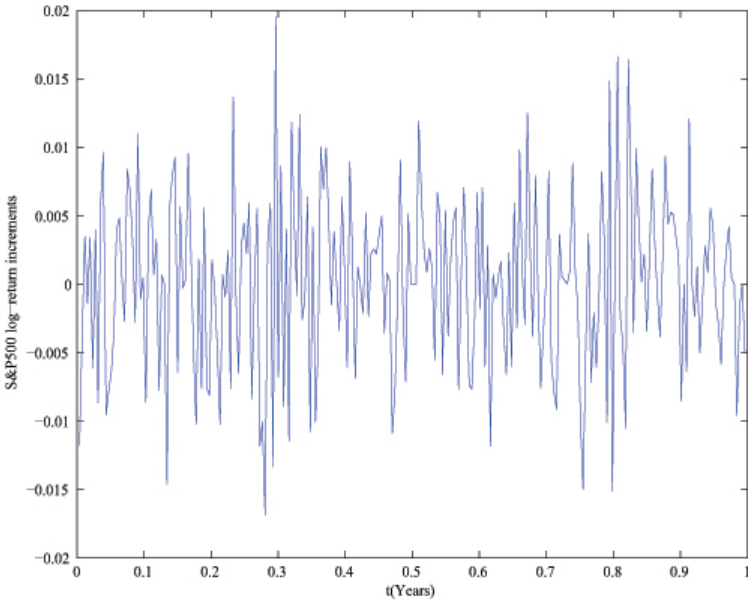
Since there are 253 trading days in the year 2005 we choose as time unit a “year” made of 253 trading days. We choose as time $t = t_0 = 0$ the day January 3, 2005.

3.1.10 The *S&P500* Index (year 2005)



We have 253 daily *S&P500* index values \hat{S}_i observed at time $t = t_i = i\Delta t$, $i = 0, 1, \dots, 252$, with $\Delta t = \frac{1}{253}$ year. Remind that $t = t_0 = 0$ corresponds to January 3, 2005.

3.1.11 The *S&P500* Daily Log-Return Increments (Year 2005)



We interpret this set of data using the Black-Scholes model.

We begin studying the variance and the drift of the Black-Scholes model used to explain *S&P500* index during the year 2005.

The *S&P500* daily log-return increments $\hat{x}_i = \ln\left(\frac{\hat{S}_i}{\hat{S}_{i-1}}\right)$, $i = 1, 2, \dots, 252$, are analyzed using the Black-Scholes model, that is they are considered as a sample of 252 observations taken from a set of i.i.d. Gaussian random variables, that is:

$$\hat{x}_i \text{ is sampled from } X_{t_i} \sim \mathcal{N}(M, V^2), i = 1, 2, \dots, 252,$$

where:

$$M = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t, V^2 = \sigma^2 \Delta t.$$

First we estimate V^2 (and therefore σ^2) using the χ^2 test and subsequently we estimate M (and therefore μ) using the Student's T test.

In order to estimate V^2 from the log-return increments we proceed as follows. Given the data sample $\hat{x}_i, i = 1, 2, \dots, n$, we fix a statistical significance level $\alpha, 0 < \alpha < 1$. We choose a sufficiently large interval $I = I^{(0)} = [a^{(0)}, b^{(0)}], 0 < a^{(0)} < b^{(0)}$ so that we can assume that $V^2 \in I^{(0)}$. We take a partition of $I^{(0)}$ made of m subintervals (of equal length) $I_i^{(0)} = [a_i^{(0)}, b_i^{(0)}], i = 1, 2, \dots, m$, and we apply the χ^2 test to test the hypothesis $V^2 \in I_i^{(0)}, i = 1, 2, \dots, m$. We restrict our attention to the subinterval(s) $I_{i^*}^{(0)} = [a_{i^*}^{(0)}, b_{i^*}^{(0)}] \subset I^{(0)}$ where the composite hypothesis:

$$H_0 : a_{i^*}^{(0)} \leq V^2 \leq b_{i^*}^{(0)}, \quad (1)$$

is accepted with significance level $\alpha, 0 < \alpha < 1$. If there are no subintervals $I_{i^*}^{(0)}$ where the hypothesis (1) is accepted we change the choice of $I^{(0)}$ and/or of m . If the subinterval $I_{i^*}^{(0)}$ is unique we set $I^{(1)} = [a^{(1)}, b^{(1)}] = I_{i^*}^{(0)}$, otherwise we set $I^{(1)} = [a^{(1)}, b^{(1)}]$ equal to the union of the intervals where (1) is accepted with significance level α , and in both cases we repeat the procedure dividing $I^{(1)}$ in the first case and shrinking $I^{(1)}$ in the second case. In this way we construct a sequence of subintervals $I^{(k)} = [a^{(k)}, b^{(k)}], k = 1, 2, \dots$, such that the hypothesis:

$$H_0 : a^{(k)} \leq V^2 \leq b^{(k)}, \quad k = 1, 2, \dots,$$

is accepted with significance level α , $0 < \alpha < 1$.

This procedure stops when $b^{(k)} - a^{(k)} < tol$, where tol is a given tolerance. We take the last set constructed with this procedure where the hypothesis formulated is accepted as final estimate of the interval where V^2 belongs with significance level α .

A similar procedure is used to estimate from the data sample \hat{x}_i , $i = 1, 2, \dots, n$, using the Student's T test, an interval where M belongs with significance level α .

Given the data sample made of the *S&P500* daily log-return increments \hat{x}_i , $i = 1, 2, \dots, 252$, the significance level $\alpha = 0.1$, $m = 2$, $tol = 10^{-4}$ and appropriate initial intervals $I = I^{(0)}$ to determine the intervals where the variance and the mean belong, we find using the procedures described above that the hypotheses:

$$\begin{aligned} 2.5297 \cdot 10^{-3} &= \sigma_1^2 \leq \sigma^2 \leq \sigma_2^2 = 2.7232 \cdot 10^{-2}, \\ -1.1087 \cdot 10^{-2} &= \mu_1 \leq \mu \leq \mu_2 = 2.5968 \cdot 10^{-2}, \end{aligned}$$

are accepted with significance level $\alpha = 0.1$.

Let us do a kind of stability analysis of the intervals determined with the statistical tests starting from the sample of the 252 daily *S&P500* log-return increments observed in the year 2005.

To do this we fix $\alpha = 0.1$ and we apply the previous procedures to determine the intervals where σ^2 and μ belong with significance level α starting from a window of 70 consecutive observations corresponding to 70 consecutive observation times (i.e. 70 consecutive trading days).

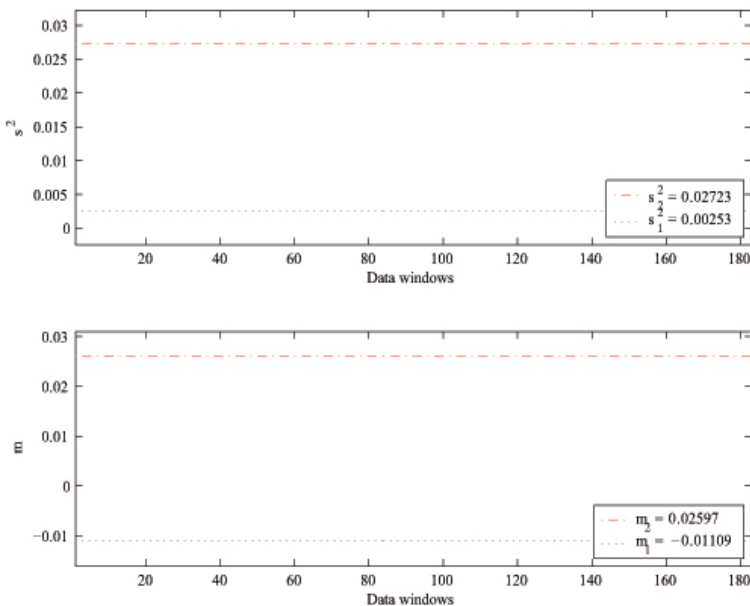
We move this window through the data time series discarding the datum corresponding to the first observation time of the window and inserting the

datum corresponding to the next observation time after the window.

Proceeding in this way we have $252-70+1$ data windows in the data time series considered and for each one of these data windows we solve the corresponding calibration problem. We find $252-70+1$ couples of intervals where the volatility and the drift parameter belong with significance level $\alpha = 0.1$.

The following figure shows that, changing the data window, the intervals determined through the statistical tests remain stable.

3.1.12 The Parameters σ^2 and μ Reconstructed from the 2005 S&P500 Data



2. Option pricing problem with uncertain volatility and statistical significance: the S&P500 call and put option prices

From the estimates:

$$\sigma_1^2 \leq \sigma^2 \leq \sigma_2^2, \quad (2)$$

$$\mu_1 \leq \mu \leq \mu_2, \quad (3)$$

established with significance level $\alpha = 0.1$, we are able to determine the corresponding option price intervals using the BSB equations. That is we can determine the worst-case option value \mathcal{V}_1 and the best-case option value \mathcal{V}_2 such that:

$$“\mathcal{V}_1(S, t) \leq \mathcal{V}(S, t) \leq \mathcal{V}_2(S, t), \quad S > 0, 0 < t < \mathcal{T} \quad (4)$$

with significance level $\alpha”$.

Note that in order to determine \mathcal{V}_1 and \mathcal{V}_2 in the Black-Scholes formula we use (2) and we choose $r = \frac{\mu_1 + \mu_2}{2}$ with μ_1 and μ_2 defined in (3). Note that the value of r (the risk free interest rate) is not really relevant in determining \mathcal{V}_1 and \mathcal{V}_2 . Moreover let us mention that in the Black-Scholes formula the time to maturity involved in the option prices is computed considering a year made of 365 days.

We compute the percentage of the prices of the European call (% call) and put (% put) options on the *S&P500* observed in the year 2005 that verify (4) when we assume (2).

The results obtained are shown in the following tables where N_{call} and N_{put} denote respectively the number of the call prices and of the put prices corresponding to options whose characteristics are described in the caption of the table.

The quantities I_{call} and I_{put} denote respectively the average relative amplitude of the call price intervals and of the put price intervals determined using the BSB equations.

P_{call} and P_{put} denote respectively the average bid price of the call and of the put prices.

In parentheses in the % call and % put columns it is written the average number of contracts on the options considered traded.

We recall that given the asset price S and the strike price K of an option, a *call* option (a *put* option) is:

1. *in the money* if $S > K$ (if $S < K$);
2. *out the money* if $S < K$ (if $S > K$);
3. *at the money* if $S = K$.

In the numerical experiments the condition $S = K$ is substituted with $|S - K| < \epsilon$ where ϵ is a (given) positive quantity. As a consequence the conditions $S > K$, $S < K$ are rewritten as $S > K + \epsilon$, and $S < K - \epsilon$, respectively. We take ϵ equal to one per cent of the average strike price of the options considered.

Using this criterion the 46.823 option prices considered above are divided in three subsets corresponding to prices of in the money, at the money and out of the money options. Table 1 refers to in the money $S\&P500$ option prices and it is obtained specifying (2), (3) respectively as:

$$2.5297 \cdot 10^{-3} = \sigma_1^2 \leq \sigma^2 \leq \sigma_2^2 = 2.7232 \cdot 10^{-2}, \quad (5)$$

$$-1.1087 \cdot 10^{-2} = \mu_1 \leq \mu \leq \mu_2 = 2.5968 \cdot 10^{-2}, \quad (6)$$

Table 1. *S&P500 option prices: in the money options (year 2005). These results are obtained using estimates (5), (6).*

January-April 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
73.8 % (172.22)	72.3 % (313.01)	1571	1822	0.23	0.34	94.01	76.82
May-August 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
74.6 % (335.25)	62.0 % (202.34)	2005	1745	0.24	0.35	92.03	76.73
September-December 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
65.5 % (401.78)	59.7 % (557.22)	2174	2300	0.23	0.35	97.94	76.14

A similar analysis relative to *S&P500* option prices corresponding to options out and at the money shows that the use of the intervals (5), (6) leads to huge call and put price intervals making the results obtained of dubious practical value.

One way of overcoming this drawback is to refine the estimates (5), (6) reducing the parameter *tol* in the procedures described previously until option price intervals of “acceptable average relative amplitude” (i.e. average relative amplitude of some tens of percentage points) are obtained.

Taking $tol_1 = tol/4 = 10^{-4}/4$ we find that the hypotheses:

$$8.7051 \cdot 10^{-3} = \sigma_1^2 \leq \sigma^2 \leq \sigma_2^2 = 1.4881 \cdot 10^{-2}, \quad (7)$$

$$1.2646 \cdot 10^{-3} = \mu_1 \leq \mu \leq \mu_2 = 1.0528 \cdot 10^{-2}, \quad (8)$$

are accepted with significance level $\alpha = 0.1$.

The choice of (7), (8) as intervals containing σ^2 and μ respectively leads to average relative amplitudes of some tens of percentage points for the option price intervals when call and put options at the money are considered.

Table 2 shows the results obtained on at the money option prices using (7), (8).

Table 2. *S&P500 option prices: at the money options (year 2005). These results are obtained using estimates (7), (8).*

January-April 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
51.1 % (1358.18)	67.7 % (1869.60)	852	902	0.27	0.28	28.29	24.76
May-August 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
56.5 % (1899.99)	66.2 % (1989.07)	1115	1154	0.28	0.28	27.40	22.52
September-December 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
41.5 % (2377.19)	59.4 % (2652.84)	1188	1208	0.27	0.28	30.29	23.45

When *S&P500* option prices relative to options out the money are considered it is necessary to reduce further the parameters *tol* to keep the

average relative amplitude of the option price intervals to reasonable values.

For example taking $tol_2 = tol/10 = 10^{-4}/10$ we find that the hypotheses:

$$1.1021 \cdot 10^{-2} = \sigma_1^2 \leq \sigma^2 \leq \sigma_2^2 = 1.2565 \cdot 10^{-2}, \quad (9)$$

$$4.7385 \cdot 10^{-3} = \mu_1 \leq \mu \leq \mu_2 = 7.0544 \cdot 10^{-3}, \quad (10)$$

are accepted with significance level $\alpha = 0.1$.

Table 3 shows the results obtained on out of the money option prices using (9), (10).

Table 3. *S&P500 option prices: out of the money options (year 2005). These results are obtained using estimates (9), (10).*

January-April 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
20.1% (691.52)	2.41 % (1061.76)	3412	4892	0.26	0.57	11.13	9.05
May-August 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
13.4% (929.68)	3.55 % (1337.39)	3644	6316	0.23	0.58	9.89	8.33
September-December 2005							
% call	% put	N_{call}	N_{put}	I_{call}	I_{put}	P_{call}	P_{put}
12.6% (1598.65)	2.71 % (1908.66)	4055	6468	0.24	0.59	12.73	8.58

3.1.13 References

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Section 3.2

The Use of Statistical Tests to Calibrate the Normal SABR Model I

[Description] We investigate the idea of solving calibration problems for stochastic dynamical systems using statistical tests. We consider a specific stochastic dynamical system: the normal SABR model. This model is a system of two stochastic differential equations whose independent variable is time and whose dependent variables are the forward prices/rates and the associated stochastic volatility. The normal SABR model is a special case of the SABR model. The calibration problem for the normal SABR model is an inverse problem that consists in determining the values of the parameters of the model from a set of data. We consider as set of data two different sets of forward prices/rates and we study the resulting calibration problems. The first set of data considered is obtained taking one observation on each trajectory of a set of independent trajectories of the normal SABR model. In the slides contained in this section the calibration problem associated to this set of data is illustrated. Ad hoc statistical tests are developed to solve this calibration problem. Estimates with statistical significance of the parameters of the model are obtained. The slides contained in this section are concerned with this calibration problem.

[Paper] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2013). The use of statistical tests to calibrate the normal SABR model, Journal of Inverse and Ill Posed Problems 21(1), 59-84.

[Website] <http://www.econ.univpm.it/recchioni/finance/w15>

3.2.1 Outline of the Presentation

- We investigate the idea of solving calibration problems for stochastic dynamical systems using statistical tests.
- We concentrate our attention on two specific stochastic dynamical systems used in mathematical finance: the Black-Scholes model (Black et al. 1973) and the normal SABR model (Hagan et al. 2002). The normal SABR model is a special case of the SABR model (Hagan et al. 2002). The calibration problem consists in determining the values of the parameters of the model starting from a set of observed data. The observed data are sets of asset prices.
- We use statistical tests to solve the calibration problem. That is we use “hypothesis testing” to determine the values of the parameters of the models. In particular to the parameter values obtained as solution of the calibration problem we associate a statistical significance level.
- We review some preliminary facts.
- We use statistical tests to calibrate the Black–Scholes asset dynamics model (Fatone et al. 2012).
- We interpret the normal SABR model (Fatone et al. Journal of Inverse and Ill Posed Problems 2013) as a state space model.
- We use statistical tests to calibrate the normal SABR model (Fatone et al. Journal of Inverse and Ill Posed Problems 2013).
- We present some numerical experiments on a sample of synthetic data.

3.2.2 Statistical Test Review

Let X_1, X_2, \dots, X_n be independent identically distributed (*i.i.d.*) random variables.

Let x_1, x_2, \dots, x_n be realizations of respectively X_1, X_2, \dots, X_n . The set x_1, x_2, \dots, x_n is the data sample of “observations”.

The mathematical setting used to describe the data sample represents “repeated experiments in a scientific laboratory”.

The (i.d.) random variables X_1, X_2, \dots, X_n have a (known) distribution $F(\underline{x}, \underline{\theta})$ depending on an (unknown) parameter vector $\underline{\theta}$.

Question: From the knowledge of the data sample x_1, x_2, \dots, x_n obtain information about $\underline{\theta}$.

3.2.3 Statistical Test - Hypothesis Testing

Hypothesis.

1. Null Hypothesis: $H_0: \underline{\theta} = \underline{\theta}_0, \underline{\theta}_0$ given;
2. Alternative Hypothesis: $H_1: \underline{\theta} \neq \underline{\theta}_0$.

Decision Table.

1. Reject H_0 ;
2. Do not reject H_0 .

Statistical significance $\alpha, 0 < \alpha < 1$.

The statistical significance α is the maximum probability of rejecting H_0

when H_0 is true (*Type I Error*).

False alarm probability β , $0 < \beta < 1$.

The false alarm probability β is the maximum probability of non rejecting H_0 when H_0 is false (*Type II Error*).

Decision rule.

Statistical test.

Applying the decision rule on the data sample, a decision of the Decision Table is assumed with statistical significance α and/or false alarm probability β .

3.2.4 Elementary Statistical Tests

Bernoulli random variable: p =probability of success; J. Arbuthnot, 1710 (statistical analysis of birthrates in London). Find p .

Normal random variable: $\mathcal{N}(\mu, \sigma^2)$, μ =mean, σ^2 =variance; *Student T*: W. S. Gosset, 1908 (Guinness Brewery, Dublin). Find μ when σ^2 is unknown. χ^2 : K. Pearson, 1900. Find σ^2 when μ is unknown.

Statistical tests have an impact in a wide variety of contexts. For example they have changed substantially:

1. criminal investigation;
2. paternity test;
3. evaluation of complex administrative systems (hospitals, tribunals, schools, ...);

4. detection of selective abortion practices.

Statistical tests are widely used in science and engineering.

We advocate their use in the solution of inverse problems in financial engineering.

The tests described in the previous slides are very simple. In practical situations the previous assumptions (i.i.d. random variables, elementary probability distributions,...) are not satisfied.

There is the need of developing ad hoc tests to solve “ad hoc” questions. These ad hoc tests may use numerical methods when necessary and in this case can be seen as examples of computational statistics.

In particular in order to use statistical tests to calibrate the asset dynamics models of mathematical finance it is necessary to consider random variables implicitly defined by stochastic dynamical systems and data samples more general than those obtained from the i.i.d. random variables.

3.2.5 Black-Scholes vs Normal SABR Model

- For the Black-Scholes model the data considered are the observations on a discrete set of time values of the asset price and the resulting calibration problem is reduced to the Student's T and the χ^2 tests (Johnson et al. 2006).
- For the normal SABR model multiple independent trajectories of the model are considered and the set of the forward prices/rates observed at a given time T in these trajectories is used as data set of the calibration problem. In the study of the normal SABR model no

elementary statistical tests (such as the Student's T and the χ^2 tests) can be used. The statistical test used to solve the calibration problem considered is based on some new formulae for the moments of the state variables of the normal SABR model and on statistical simulation.

Remarks

1. The calibration problems studied are *inverse problems* for stochastic dynamical systems.
2. The results presented for the normal SABR model are easily extended to several other contexts in science and engineering where similar stochastic models are used. These are models involving stochastic volatility or, more in general, stochastic state space models. For example models of this type are used in the study of clutter in signal processing, of wave propagation in random acoustic or electromagnetic media, of noise in telecommunications, of biomedical systems in medicine, as well as in the study of other models in mathematical finance.
3. The solution of calibration problems in mathematical finance usually does not involve statistical tests and statistical significance levels. That is to the solution of the calibration problem found is not attributed a "confidence level".

3.2.6 The Black-Scholes Model

Let $S_t > 0$ denote the asset price at time $t \geq 0$. The Black-Scholes model (Black et al. 1973) assumes that $S_t, t > 0$, satisfies:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t > 0,$$

$$S_0 = \hat{S}_0,$$

where μ, σ are real parameters, μ is the drift, $\sigma > 0$ is the volatility, $W_t, t > 0$, is a standard Wiener process such that $W_0 = 0, dW_t, t > 0$, is its stochastic differential and $\hat{S}_0 > 0$ is a given random variable. We assume \hat{S}_0 concentrated in a point with probability one.

The real parameters μ, σ are the unknowns of the calibration problem for the Black- Scholes model.

Let $\Delta t > 0$ be a time increment and $t_i = i\Delta t, i = 0, 1, \dots, n$, be a discrete set of equispaced time values. We define X_{t_i} , the asset price log-return increment when t goes from t_{i-1} to $t_i, i = 1, 2, \dots, n$, as follows:

$$X_{t_i} = \ln \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right), \quad i = 1, 2, \dots, n.$$

The random variables $X_{t_i}, i = 1, 2, \dots, n$, are independent identically distributed (i.i.d.) Gaussian random variables with mean M and variance V^2 where $M = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t, V^2 = \sigma^2 \Delta t$. That is we have:

$$X_{t_i} \sim \mathcal{N} \left(\left(\mu - \frac{\sigma^2}{2} \right) \Delta t, \sigma^2 \Delta t \right), \quad i = 1, 2, \dots, n.$$

3.2.7 The Black-Scholes Calibration Problem

Given $\Delta t > 0$, a statistical significance level $\alpha, 0 < \alpha < 1$, and the asset price \hat{S}_i observed at time $t = t_i = i\Delta t, i = 0, 1, \dots, n$, determine two

intervals where the parameters of the Black-Scholes model μ and $\sigma > 0$ belong with the given significance level α .

Given a significance level α , $0 < \alpha < 1$, we can perform statistical tests on the variance V^2 and on the mean M of the random variables $X_{t_i} = \ln(S_{t_i}/S_{t_{i-1}})$, $i = 1, 2, \dots, n$, starting from the data sample $\hat{x}_i = \ln(\hat{S}_i/\hat{S}_{i-1})$, $i = 1, 2, \dots, n$, using the χ^2 test and the Student's T test respectively (Johnson et al. 2006).

This implies that given α , $0 < \alpha < 1$, we can accept or reject, with significance level α , the null hypothesis:

$$H_0 : \sigma_1 \leq \sigma \leq \sigma_2, \quad \text{or} \quad H_0 : \mu_1 \leq \mu \leq \mu_2,$$

where

$$\sigma_i = \frac{V_i}{\sqrt{\Delta t}}, \quad \text{and} \quad \mu_i = \frac{M_i}{\Delta t} + \frac{V_i^2}{2\Delta t}, \quad i = 1, 2,$$

simply translating to σ and μ the results on V^2 and M obtained with the Student's T and the χ^2 tests (Fatone et al. 2012).

Remark

Given a significance level α , $0 < \alpha < 1$, let us assume that the hypothesis $H_0 : \sigma_1 \leq \sigma \leq \sigma_2$ is accepted with significance level α . Using the Black-Scholes- Barenblatt equation it is possible to determine the corresponding range where the value of an European vanilla option lies (with significance level α) (see Fatone et al. 2012).

3.2.8 The Normal SABR Model

Let ξ_t , v_t , $t > 0$, be real stochastic processes that describe respectively the forward prices/rates and the associated stochastic volatility. The normal SABR model is given by Hagan et al. 2002 :

$$d\xi_t = v_t dW_t, \quad t > 0, \quad \xi_0 = \tilde{\xi}_0,$$

$$dv_t = \varepsilon v_t dQ_t, \quad t > 0, \quad v_0 = \tilde{v}_0.$$

The quantity $\varepsilon > 0$ is a parameter known as volatility of volatility. The stochastic processes $W_t, Q_t, t > 0$, are standard Wiener processes such that $W_0 = Q_0 = 0, dW_t, dQ_t, t > 0$, are their stochastic differentials and we assume that:

$$\langle dW_t dQ_t \rangle = \rho dt, \quad t > 0,$$

where $\langle \cdot \rangle$ denotes the expected value of \cdot and $\rho \in (-1, 1)$ is a constant known as correlation coefficient. The quantities $\tilde{\xi}_0, \tilde{v}_0$ are random variables that we assume to be concentrated in a point with probability one.

We assume $\tilde{v}_0 > 0$. Unlike $\tilde{\xi}_0$ the initial stochastic volatility \tilde{v}_0 cannot be observed in the financial markets and must be regarded as a parameter of the model. Similarly the stochastic volatility $v_t, t > 0$, cannot be observed in the financial markets.

The unknowns of the calibration problem for the normal SABR model are: $\varepsilon, \rho, \tilde{v}_0$.

The normal SABR model and more in general the SABR model are widely used in the pricing of interest rates derivatives and of options on currencies exchange rates in the theory and practice of mathematical finance.

3.2.9 The Normal SABR Model Interpreted as a State Space Model

Interpreting $v_t, t > 0$, as state variable (not observed) and $\xi_t, t > 0$, as observation variable, the normal SABR model:

$$dv_t = \varepsilon v_t dQ_t, \quad t > 0, \text{ state(or transition) equation,}$$

$$d\xi_t = v_t dW_t, \quad t > 0, \text{ observation(or measurement) equation,}$$

becomes a (stochastic) state space model.

Remark

In a similar way it is easy to see that the usual stochastic volatility models used in mathematical finance (Heston, Hull and White, Stein and Stein, ...) can be interpreted as (stochastic) state space models.

Remark

The state space models have been introduced around 1960 by Kalman in the study of guidance problems in the aeronautical industry. Today they are widely used in many branches of engineering (in particular in signal processing) including in financial engineering (i.e. stochastic volatility models).

Recall that:

1. The real parameters $\varepsilon, \rho, \tilde{v}_0$, are the unknowns of the calibration problem for the normal SABR model.
2. We use a statistical test to solve a calibration problem for the normal SABR model; more precisely to the parameter values obtained as solution of the calibration problem we associate a statistical

significance level using an ad hoc statistical test. The statistical test used is based on some new formulae for the moments of the forward prices/rates variable of the normal SABR model. In particular the decision resulting from the test is assumed comparing the theoretical values of three moments of the forward prices/rates variable of the normal SABR model when the null hypothesis H_0 is true with the observed values of these moments computed from the data sample.

In Fatone et al. Journal of Mathematical Finance 2013 we derive the following formula for the transition probability density function p_N of the variables $\xi_t, v_t, t > 0$, implicitly defined by the normal SABR model:

$$p_N(\xi, v, t, \xi', v', t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{-\imath k(\xi' - \xi)} g_N(t - t', k, v, v', \varepsilon, \rho),$$

$$(\xi, v), (\xi', v') \in \mathbb{R} \times \mathbb{R}^+, t, t' \geq 0, t - t' > 0,$$

where $\xi_t = \xi, v_t = v, \xi_{t'} = \xi', v_{t'} = v', t, t' \geq 0, t - t' > 0$. The function g_N is given by:

$$g_N(s, k, v, v', \varepsilon, \rho) =$$

$$\frac{2}{\pi^2} e^{-\frac{s}{8}\varepsilon^2} \left(\frac{\sqrt{v'}}{v\sqrt{v}} \right) e^{\imath k \frac{\rho(v'-v)}{\varepsilon}} \int_0^{+\infty} d\omega e^{-\tau\omega^2} \omega \sinh(\pi\omega) K_{\imath\omega}(\zeta(k)v) K_{\imath\omega}(\zeta(k)v'),$$

$$s \in \mathbb{R}^+, k \in \mathbb{R}, v, v' \in \mathbb{R}^+, \varepsilon > 0, \rho \in (-1, 1),$$

where \imath is the imaginary unit, the functions \sinh, K_η denote respectively the hyperbolic sine and the second type modified Bessel function of order η and finally $\zeta^2(k), k \in \mathbb{R}$, is defined as: $\zeta^2(k) = \frac{k^2}{\varepsilon^2} (1 - \rho^2), k \in \mathbb{R}$.

Starting from the previous formula in Fatone et al. *Journal of Inverse and Ill Posed Problems* 2013 we derive the formulae for the moments $\mathcal{M}_{n,m}$, $n, m = 0, 1, \dots$, with respect to zero of the transition probability density function p_N , that is:

$$\mathcal{M}_{n,m}(t, \xi', v', t') = \int_{-\infty}^{+\infty} d\xi \xi^n \int_0^{+\infty} dv v^m p_N(\xi, v, t, \xi', v', t'),$$

$$(\xi', v') \in \mathbb{R} \times \mathbb{R}^+, t, t' \geq 0, t - t' > 0, n, m = 0, 1, \dots$$

Let

$$\mathcal{M}_n^*(t - t', \xi', v') = \mathcal{M}_{n,0}(t, \xi', v', t'),$$

$$t - t' \in \mathbb{R}^+, (\xi', v') \in \mathbb{R} \times \mathbb{R}^+, n = 0, 1, \dots$$

Note that the moments \mathcal{M}_n^* when $t' = 0, \xi' = \tilde{\xi}_0, v' = \tilde{v}_0$ depend on the unknowns of the calibration problem considered $\varepsilon, \rho, \tilde{v}_0$ and on the time t . In particular \mathcal{M}_2^* depends on ε and \tilde{v}_0 , while the moments \mathcal{M}_3^* and \mathcal{M}_4^* depend on ε, \tilde{v}_0 and ρ .

In Fatone et al. *Journal of Inverse and Ill Posed Problems* 2013 when $t' = 0, \xi' = \tilde{\xi}_0, v' = \tilde{v}_0$ we derive the following new closed form formulae for the moments $\mathcal{M}_0^*, \mathcal{M}_1^*, \mathcal{M}_2^*, \mathcal{M}_3^*, \mathcal{M}_4^*$ of the normal SABR model:

$$\mathcal{M}_0^*(t, \tilde{\xi}_0, \tilde{v}_0) = 1, \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\mathcal{M}_1^*(t, \tilde{\xi}_0, \tilde{v}_0) = \tilde{\xi}_0, \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\mathcal{M}_2^*(t, \tilde{\xi}_0, \tilde{v}_0) = \tilde{\xi}_0^2 + \frac{\tilde{v}_0^2}{\varepsilon^2}(e^{\varepsilon^2 t} - 1), \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\mathcal{M}_0^*(t, \tilde{\xi}_0, \tilde{v}_0) = 1, \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\mathcal{M}_1^*(t, \tilde{\xi}_0, \tilde{v}_0) = \tilde{\xi}_0, \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\mathcal{M}_2^*(t, \tilde{\xi}_0, \tilde{v}_0) = \tilde{\xi}_0^2 + \frac{\tilde{v}_0^2}{\varepsilon^2}(e^{\varepsilon^2 t} - 1), \quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+,$$

$$\begin{aligned} \mathcal{M}_3^*(t, \tilde{\xi}_0, \tilde{v}_0) &= \tilde{\xi}_0^3 + 3\tilde{\xi}_0 \frac{\tilde{v}_0^2}{\varepsilon^2}(e^{\varepsilon^2 t} - 1) \\ &\quad + \frac{\rho \tilde{v}_0^3}{\varepsilon^3} e^{3\varepsilon^2 t} \left(1 - 3e^{-2\varepsilon^2 t} + 2e^{-3\varepsilon^2 t} \right), \\ &\quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+, \end{aligned}$$

$$\begin{aligned} \mathcal{M}_4^*(t, \tilde{\xi}_0, \tilde{v}_0) &= \tilde{\xi}_0^4 + 6\tilde{\xi}_0^2 \frac{\tilde{v}_0^2}{\varepsilon^2}(e^{\varepsilon^2 t} - 1) \\ &\quad - 4\tilde{\xi}_0 \frac{\rho \tilde{v}_0^3}{\varepsilon^3} e^{3\varepsilon^2 t} \left(1 - 3e^{-2\varepsilon^2 t} + 2e^{-3\varepsilon^2 t} \right) \\ &\quad + \frac{(\tilde{v}_0)^4}{\varepsilon^4} e^{6t\varepsilon^2} \left[\frac{4\rho^2}{3}(1 - e^{-3\varepsilon^2 t}) + \right. \\ &\quad \left. \frac{(6 - 12\rho^2)}{5}(1 - e^{-5\varepsilon^2 t}) + \frac{(8\rho^2 - 6)}{6}(1 - e^{-6\varepsilon^2 t}) \right], \\ &\quad (\tilde{\xi}_0, \tilde{v}_0) \in \mathbb{R} \times \mathbb{R}^+, \quad t \in \mathbb{R}^+. \end{aligned}$$

3.2.10 The Normal SABR Calibration Problem

Given a time value $T > 0$, a statistical significance level α , $0 < \alpha <$

1, a positive integer n and n independent observations at time $t = T$ of the forward prices/rates ξ_T , that is given $\hat{\xi}_T^i$, $i = 1, 2, \dots, n$, determine the values of the parameters ε , ρ and \tilde{v}_0 of the normal SABR model with significance level α .

Let $T > 0$ be given and let ξ_T^i , $i = 1, 2, \dots, n$, be n independent copies of the random variable ξ_T . It is easy to see that the random variables:

$$\bar{X}(n, T) = \frac{1}{n} \sum_{i=1}^n X_i, \bar{Y}(n, T) = \frac{1}{n} \sum_{i=1}^n Y_i, \bar{Z}(n, T) = \frac{1}{n} \sum_{i=1}^n Z_i,$$

where $X_i = (\xi_T^i)^2$, $Y_i = (\xi_T^i)^3$, $Z_i = (\xi_T^i)^4$, $i = 1, 2, \dots, n$, are unbiased estimators of respectively \mathcal{M}_2^* , \mathcal{M}_3^* , \mathcal{M}_4^* (when $t = T$, $t' = 0$, $\xi' = \tilde{\xi}_0$, $v' = \tilde{v}_0$). The random variables \bar{X} , \bar{Y} , \bar{Z} are used as components of the vector valued test statistic $L = (\bar{X}, \bar{Y}, \bar{Z})$ of the statistical test. Note that for $i = 1, 2, \dots, n$, the observation $\hat{\xi}_T^i$ can be regarded as a realization of the random variable ξ_T^i .

Let us consider the realizations $\hat{X}, \hat{Y}, \hat{Z}$ in the data sample $\hat{\xi}_T^i$, $i = 1, 2, \dots, n$, of the random variables $\bar{X}, \bar{Y}, \bar{Z}$, that is:

$$\hat{X}(n, T) = \frac{1}{n} \sum_{i=1}^n \hat{X}_i, \hat{Y}(n, T) = \frac{1}{n} \sum_{i=1}^n \hat{Y}_i, \hat{Z}(n, T) = \frac{1}{n} \sum_{i=1}^n \hat{Z}_i,$$

where $\hat{X}_i = (\hat{\xi}_T^i)^2$, $\hat{Y}_i = (\hat{\xi}_T^i)^3$, $\hat{Z}_i = (\hat{\xi}_T^i)^4$, $i = 1, 2, \dots, n$.

Given a statistical significance level α , $0 < \alpha < 1$, and given $\varepsilon^* > 0$, $\rho^* \in (-1, 1)$, $\tilde{v}_0^* > 0$, using the vector valued test statistic $L = (\bar{X}, \bar{Y}, \bar{Z})$, we want to test the null hypothesis:

$$H_0 : (\varepsilon, \rho, \tilde{v}_0) = (\varepsilon^*, \rho^*, \tilde{v}_0^*).$$

with statistical significance level α , $0 < \alpha < 1$.

First of all we translate the hypothesis H_0 in a corresponding hypothesis for the moments $\mathcal{M}_2^*, \mathcal{M}_3^*, \mathcal{M}_4^*$ associated to the normal SABR model.

The moments $\hat{\mathcal{M}}_2^*, \hat{\mathcal{M}}_3^*, \hat{\mathcal{M}}_4^*$ obtained from $\mathcal{M}_2^*, \mathcal{M}_3^*, \mathcal{M}_4^*$ when H_0 is true and $t = T, t' = 0$, are compared with the moments $\hat{X}, \hat{Y}, \hat{Z}$ observed in the data sample. Note that the point $\hat{P} = (\hat{X}, \hat{Y}, \hat{Z}) \in \mathbb{R}^3$ is the value taken by the test statistic L on the data sample.

In particular to test the null hypothesis H_0 we check if the point $\hat{P} = (\hat{X}, \hat{Y}, \hat{Z}) \in \mathbb{R}^3$ and the point $\hat{P}^* = (\hat{\mathcal{M}}_2^*, \hat{\mathcal{M}}_3^*, \hat{\mathcal{M}}_4^*) \in \mathbb{R}^3$ are “close” or “far”.

The heuristic decision rule is:

1. do not reject H_0 if the points \hat{P} and \hat{P}^* are “close”;
2. reject H_0 if the points \hat{P} and \hat{P}^* are “far”.

In Fatone et al. Journal of Inverse and Ill Posed Problems 2013 we determine the relation among $\alpha, n, \varepsilon^*, \rho^*, \tilde{v}_0^*$ that translates the qualitative expressions “close” and “far” in a quantitative statement about the norm of the vector $\hat{P} - \hat{P}^*$.

Recall that the statistical significance level $\alpha, 0 < \alpha < 1$, is the maximum probability of rejecting the null hypothesis H_0 when the hypothesis is true.

We proceed as follows: given $\alpha, 0 < \alpha < 1, n > 0, \varepsilon^* > 0, \rho^* \in (-1, 1)$ and $\tilde{v}_0^* > 0$ we solve the following inequality for the real unknown $A_{\alpha, n}$:

$$Probability(\|L - \hat{P}^*\| \geq A_{\alpha, n}) \leq \alpha, \quad (1)$$

where $\|\cdot\|$ is the norm of \cdot in \mathbb{R}^3 and we determine the infimum $r_{\alpha, n}$ of the values $A_{\alpha, n}$ that satisfy (1).

The inequality (1) is studied and the infimum $r_{\alpha,n}$ of its solutions is determined numerically using statistical simulation.

Given α , $0 < \alpha < 1$, a positive integer n , the null hypothesis H_0 and the corresponding threshold $r_{\alpha,n} > 0$ the decision rule of the statistical test is:

1. if $\|\hat{P} - \hat{P}^*\| \leq r_{\alpha,n}$ do not reject H_0 , with significance level α ;
2. if $\|\hat{P} - \hat{P}^*\| > r_{\alpha,n}$ reject H_0 , with significance level α .

Let us call “moments space” (i.e. \mathbb{R}^3) the space where the test statistic L takes values. Note that the threshold $r_{\alpha,n}$ divides the “moments space” into two regions: the rejection region $R = R_{\alpha,n}$ and the “retain” (i.e. do not reject) region. In the “moments space” the retain region is the sphere of center the vector of the theoretical moments \hat{P}^* and radius $r_{\alpha,n}$ and the rejection region $R_{\alpha,n}$ is its complement.

3.2.11 Procedure Used to Determine the Threshold $r_{\alpha,n}$

We build a sample of the random variables $\bar{X}, \bar{Y}, \bar{Z}$ integrating numerically the normal SABR model (when H_0 is true) in the time interval $[0, T]$ with the explicit Euler method.

The joint probability density function of the random variables $\bar{X}, \bar{Y}, \bar{Z}$ is approximated with the corresponding three-dimensional joint histogram deduced from the sample of $\bar{X}, \bar{Y}, \bar{Z}$ generated.

This histogram shows the proportion (i.e. relative frequency) of cases that falls into each one of several disjoint categories covering \mathbb{R}^3 (i.e. non-overlapping three-dimensional parallelepipeds covering \mathbb{R}^3). The proportion of cases belonging to each category approximates the probability that the random variable $L = (\bar{X}, \bar{Y}, \bar{Z})$ (when H_0 is true) belong to that category. The sum of these proportions is equal to one.

Note that the random variables $\bar{X}, \bar{Y}, \bar{Z}$, their joint probability density function and therefore their joint histogram depend on H_0, T and n .

Given α, n, H_0 and the joint histogram of the variables $\bar{X}, \bar{Y}, \bar{Z}$ when H_0 is true, the threshold $r_{\alpha,n}$ is obtained integrating the joint probability density function of $\bar{X}, \bar{Y}, \bar{Z}$ on the spheres of center \hat{P}^* and radius $A_{\alpha,n}$, $A_{\alpha,n} > 0$. These integrals are approximated with appropriate sums of relative frequencies of the joint histogram of $\bar{X}, \bar{Y}, \bar{Z}$ obtained with the numerical simulation. In this way we determine $r_{\alpha,n}$ as the infimum of the $A_{\alpha,n}$ that, in the approximations considered, satisfy condition (1).

Remark

Note that the threshold $r_{\alpha,n}$ depends on α and n ; moreover unlike the threshold(s) of the elementary statistical tests of the normal random variable (i.e. the Student's T or χ^2 tests, see Johnson et al. 2006) and of the tests used in the calibration of the Black-Scholes model (Fatone et al. 2012), $r_{\alpha,n}$ depends on the null hypothesis H_0 .

In the study case that follows given n and the null hypothesis H_0 we provide a table of $r_{\alpha,n}$ as a function of $\alpha, 0 < \alpha < 1$.

3.2.12 A Drawback

The data sample used in the previous statistical test although realistic in many contexts of science and engineering it is hardly available in the financial markets. In fact in the financial markets usually it is not possible to repeat the “experiment”, that is repeated observations at time $t = T$ of independent trajectories of the stochastic dynamical system under investigation are usually not available. This is a serious drawback.

In Fatone at al. Journal of Inverse and Ill Posed Problems 2013 and in

Section 3.3 of Chapter 3 it is presented a second statistical test to calibrate the normal SABR model that uses a different data sample. The data sample used by the second statistical test is easily available in the financial markets.

Remarks

1. Several null hypotheses different from $H_0 : (\varepsilon, \rho, \tilde{v}_0) = (\varepsilon^*, \rho^*, \tilde{v}_0^*)$ can be studied adapting the statistical test considered here.
2. In Fatone et al. 2012 and Fatone et al. Journal of Inverse and Ill Posed Problems 2013 we discuss the question of how to choose the parameters that defines the null hypothesis H_0 to be tested in the calibration problems for the Black-Scholes and for the normal SABR models respectively. For example these parameters can be chosen as solution of a different formulation of the calibration problem that does not involve statistical significance.
3. The statistical test used to calibrate the normal SABR model uses numerical methods (i.e. the probability density function of the test statistic is approximated by statistical simulation) and can be seen as an example of a fruitful application of numerical methods in statistics, that is it is an example of computational statistics.

3.2.13 A Numerical Example

We solve the calibration problem for the normal SABR model with the statistical test procedure presented using a sample of synthetic data.

Let $T > 0$ be given and n, m be positive integers. Let $\Delta t = T/m$ be a time increment and $t_i = i\Delta t, i = 0, 1, \dots, m$, be a discrete set of

equispaced time values. Let $\xi_{t_m} = \xi_T$, $v_{t_m} = v_T$ be the solutions of the normal SABR model at time $t = T$.

We approximate n independent realizations $\hat{\xi}_T^i$, $i = 1, 2, \dots, n$, of the random variable ξ_T integrating numerically n times the normal SABR model in the time interval $[0, T]$ using the explicit Euler method.

We choose $T = 1$, $m = 10000$, $n = 100$, $\varepsilon = 0.1$, $\rho = -0.2$, $\xi_0 = \tilde{\xi}_0 = 0$ and $v_0 = \tilde{v}_0 = 0.5$. That is:

$$(\varepsilon, \rho, \tilde{v}_0) = (0.1, -0.2, 0.5),$$

are the unknown parameters of the normal SABR model that we want to recover as solution of in the calibration problem.

The synthetic data $\hat{\xi}_{T=1}^i$, $i = 1, 2, \dots, n$, are obtained approximating with the explicit Euler method multiple independent trajectories of the normal SABR model with the previous parameter values and looking at the computed trajectories at time $t = T = 1$.

That is for $n = 100$ and $i = 1, 2, \dots, n$, let $\hat{\xi}_{T=1}^i$ be the approximation of $\xi_{T=1}^i$ obtained in this way.

The set $\bar{D}_{T=1} = \left\{ \hat{\xi}_{T=1}^i, i = 1, 2, \dots, 100 \right\}$ is the data sample of the statistical test used in the calibration problem of the normal SABR model.

We consider the following calibration problem:

Given $\bar{D}_{T=1}$ and the significance level α , $0 < \alpha < 1$, determine the values of the parameters $(\varepsilon, \rho, \tilde{v}_0)$ of the normal SABR model with significance level α .

- The first step consists in the formulation of the null hypothesis H_0 . We proceed as suggested in Fatone et al. Journal of Inverse and Ill

Posed Problems 2013 and we solve the calibration problem using the nonlinear least squares method. As a result of this analysis we test the null hypothesis:

$$\bar{H}_0 : (\varepsilon, \rho, \tilde{v}_0) = (\varepsilon^*, \rho^*, \tilde{v}_0^*) = (0.1261, -0.3356, 0.515),$$

with statistical significance level α using the data sample $\bar{D}_{T=1}$.

- To perform this test the corresponding threshold $r_{\alpha,100}$ must be determined.

For this purpose we build a sample of $N = 1000$ (approximate) realizations of the random variables $\bar{X}, \bar{Y}, \bar{Z}$ defined previously when $n = 100$, $T = 1$ and \bar{H}_0 is true integrating numerically (100000 times) with the explicit Euler method the normal SABR model (when \bar{H}_0 is true) in the time interval $[0, 1]$.

Moreover we approximate the joint probability density function of the previously defined random variables $\bar{X}, \bar{Y}, \bar{Z}$ with the corresponding three-dimensional joint histogram associated to the sample of numerosness $N = 1000$ of the random variables $\bar{X}, \bar{Y}, \bar{Z}$ that has been generated. Proceeding as suggested in the previous slides we determine an approximation of $r_{\alpha,100}$ denoted with $\bar{r}_{\alpha,100}$. For simplicity we consider $r_{\alpha,100} = \bar{r}_{\alpha,100}$.

Given $n = 100$ and the null hypothesis \bar{H}_0 table 4 shows the values of the the threshold $r_{\alpha,100} = \bar{r}_{\alpha,100}$ as a function of α determined with the previous procedure.

Table 4. The threshold $r_{\alpha,n} = \bar{r}_{\alpha,100}$ as a function of α for the null hypothesis \bar{H}_0 .

$\bar{H}_0 : (\varepsilon, \rho, \tilde{v}_0) = (0.1261, -0.3356, 0.515), \quad n = 100$	
α	$\bar{r}_{\alpha,100}$
0.01	0.29
0.05	0.19
0.1	0.16

Let us perform the test associated to the calibration problem considered.

Given the null hypothesis \bar{H}_0 , the significance level α and the data sample $\bar{D}_{T=1}$ made of $n = 100$ observations of the random variable $\xi_{T=1}$ compute:

1. the point $\hat{Q} = (\hat{X}, \hat{Y}, \hat{Z}) \in \mathbb{R}^3$ associated to the data sample $\bar{D}_{T=1}$;
2. the point $\hat{Q}^* = (\hat{\mathcal{M}}_2^*, \hat{\mathcal{M}}_3^*, \hat{\mathcal{M}}_4^*) \in \mathbb{R}^3$, where the quantities $\hat{\mathcal{M}}_2^*, \hat{\mathcal{M}}_3^*, \hat{\mathcal{M}}_4^*$ are the “theoretical” moments $\mathcal{M}_2^*, \mathcal{M}_3^*, \mathcal{M}_4^*$ calculated when $t = T = 1, \varepsilon = 0.1261, \rho = -0.3356, \tilde{v}_0 = 0.515$ (i.e. are the moments $\mathcal{M}_2^*, \mathcal{M}_3^*, \mathcal{M}_4^*$ evaluated when $t = T = 1$ and the hypothesis \bar{H}_0 is true).

We have $\hat{Q} = (0.2674, -0.0177, 0.2277)$ and $\hat{Q}^* = (0.2674, -0.0177, 0.2197)$.

Let $\alpha = 0.01, 0.05, 0.1$, and $\bar{r}_{\alpha,100}$ be the corresponding thresholds shown in Table 4, the decision rule of the statistical test that has statistical significance α is given by:

1. if $\|\hat{Q} - \hat{Q}^*\| \leq \bar{r}_{\alpha,100}$ do not reject \bar{H}_0 , with significance level α ;
2. if $\|\hat{Q} - \hat{Q}^*\| > \bar{r}_{\alpha,100}$ reject \bar{H}_0 , with significance level α .

In this specific experiment given the data sample $\bar{D}_{T=1}$ the hypothesis \bar{H}_0 , is retained for the values of α considered in Table 4.

3.2.14 Future Work

- Derive closed form formulae for the moments of the lognormal SABR model and use them in the calibration of the model.
- Develop new statistical tests and apply them to sets of real data time series of forward prices/rates actually observed in the financial markets.
- Use the previous ideas to calibrate other stochastic dynamical systems used in mathematical finance.

3.2.15 References

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Section 3.3

The Use of Statistical Tests to Calibrate the Normal SABR Model II

[Description] We investigate the idea of solving calibration problems for stochastic dynamical systems using statistical tests. We consider a specific stochastic dynamical system: the normal SABR model. This model is a system of two stochastic differential equations whose independent variable is the time and whose dependent variables are the forward prices/rates and the associated stochastic volatility. The normal SABR model is a special case of the SABR model. The calibration problem for the normal SABR model is an inverse problem that consists in determining the values of the parameters of the model from a set of data. We consider as set of data two different sets of forward prices/rates and we study the resulting calibration problems. The second set of data considered is obtained taking several observations of the forward prices/rates on a discrete set of given time values along a single trajectory of the normal SABR model. Ad hoc statistical tests are developed to solve this calibration problem. Estimates with statistical significance of the parameters of the model are obtained. The slides contained in this section are concerned with this calibration problem.

[Paper] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2013). The use of statistical tests to calibrate the normal SABR model, *Journal of Inverse and Ill Posed Problems* 21(1), 59-84.

[Website] <http://www.econ.univpm.it/recchioni/finance/w15>

3.3.1 Outline of the Presentation

1. Calibration

- Normal SABR model
- Formulation of the calibration problem
- Calibration techniques

2. An ad hoc statistical test

- Hypothesis formulation
- Decision rule
- Monte Carlo method

3. Numerical Results

4. References

3.3.2 Normal SABR Model

Let us consider the normal SABR model introduced by Hagan et al. 2002. We assume that the forward price of an asset ξ_t , $t > 0$, and its stochastic volatility v_t , $t > 0$, satisfy the following system of stochastic differential equations:

$$d\xi_t = v_t dW_t, \quad t > 0, \quad (1)$$

$$dv_t = \varepsilon v_t dQ_t, \quad t > 0, \quad (2)$$

with the initial conditions:

$$\xi_0 = \tilde{\xi}_0, \quad (3)$$

$$v_0 = \tilde{v}_0. \quad (4)$$

In (1)-(4):

- $\varepsilon > 0$ is the volatility of volatility,
- $W_t, Q_t, t > 0$, are standard Wiener processes such that
 - 1) $W_0 = Q_0 = 0$,
 - 2) $dW_t, dQ_t, t > 0$, are their stochastic differentials,
 - 3) $\langle dW_t dQ_t \rangle = \rho dt, t > 0$, where $\langle \cdot \rangle$ denotes the expected value of \cdot and $\rho \in (-1, 1)$ is the correlation coefficient,
- $\tilde{\xi}_0, \tilde{v}_0$ are random variables that are assumed to be concentrated in a point with probability one,
- $\tilde{v}_0 > 0$ is not observable and must be considered as a parameter of the model.

The unknowns of the calibration problem are $\varepsilon, \rho, \tilde{v}_0$.

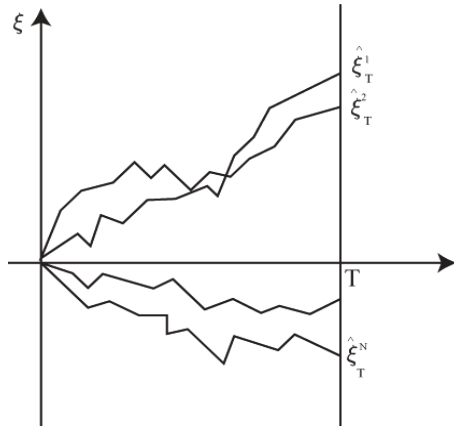
Let us formulate the calibration problem for the normal SABR model (1), (2), (3), (4) that we study.

- Let M be a positive integer. The data of the calibration problem are the forward prices/rates observed at discrete times t_0, t_1, \dots, t_M , such that $t_i > t_{i-1}, i = 1, 2, \dots, M$, where $t_0 = 0$. For $i = 1, 2, \dots, M$ we denote with $\hat{\xi}_i$ the forward price/rate observed at time $t = t_i$ along one trajectory of the stochastic process $\xi_t, t > 0$. The set $D_1 = \{\hat{\xi}_i, i=1, 2, \dots, M\}$ is the data sample used to solve the calibration problem.

Calibration problem: From the knowledge of the data sample D_1 we want to recover the values of the unknown parameters of the normal SABR model: $\varepsilon, \rho, \tilde{v}_0$.

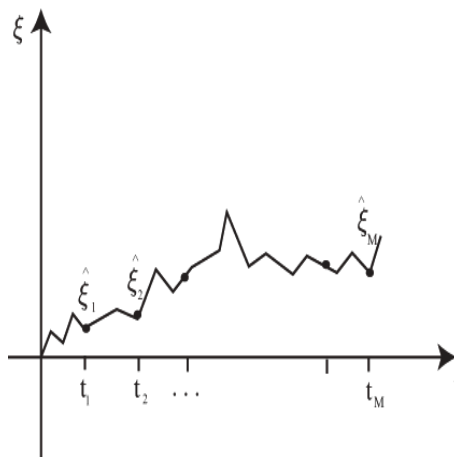
Data sample of the calibration problem

- Previous talk



Data sample 1: N observations of $\hat{\xi}_T^i$, $i = 1, 2, \dots, N$, made at time $t = T$ on N independent trajectories.

- This talk



Data sample 2: M observations made at the at times t_i , $i = 0, 1, 2, \dots, M$, on a single trajectory.

- Data sample 1 corresponds to “repeated experiments” (repetitions of a phenomenon). This data sample most of the times is not realistic in finance.
- Data sample 2 corresponds to observing the time evolution of a phenomenon. This data sample is realistic in finance. The observations are simply the time series of the prices observed.

3.3.3 Calibration Techniques

In the literature there are several methods to calibrate stochastic volatility models like model (1)-(4).

- Maximum likelihood estimation (Mariani et. al. 2008): the values of the model parameters are characterized as those that give to the observed data the greatest likelihood (i.e. parameter values that maximize the likelihood function).
- blueLeast squares fit (Fatone et al. 2008): the values of the model parameters are characterized as those that minimize the least square error between the observed data and the values predicted by the model of the observed quantities.
- Method of moments (Garcia et al. 2011): the values of the model parameters are characterized as those that minimize the difference between the moments predicted by the model and the moments obtained from the observed data.

To the solution of the calibration problem obtained with these methods is not associated a statistical significance level.

3.3.4 An ad hoc Statistical Test

We want to solve the calibration problem associating to the solution found a statistical significance level through the use of a statistical test.

- We state a null and an alternative hypothesis.
- We choose a significance level.
- We specify a decision rule.
- We compute the test statistic on the data sample.
- We take a decision. That is we reject or we do not reject the null hypothesis with the chosen significance level.

3.3.5 Hypothesis Formulation

- Starting from the data sample $D_1 = \{\hat{\xi}_i, i = 1, 2, \dots, M\}$ we solve the calibration problem using one of the techniques mentioned previously (i.e.: maximum likelihood, least squares fit, moments method). Let $\varepsilon^*, \rho^*, \tilde{v}_0^*$ be the values of the normal SABR model (1)-(4) obtained as solution of the calibration problem.
- We choose the following null hypothesis:

$$H_0 : (\varepsilon, \rho, \tilde{v}_0) = (\varepsilon^*, \rho^*, \tilde{v}_0^*), \quad (5)$$

- We choose a statistical significance level $\alpha \in (0, 1)$, i.e. $\alpha =$ maximum probability of rejecting H_0 when H_0 is true. (type I error)

- We test the null hypothesis H_0 with significance level α .

3.3.6 Decision Rule

How to decide when to reject the null hypothesis H_0 ? There are two possibilities:

- the data sample D_1 is “compatible” with (does not contradict) H_0 with significance level α : the hypothesis H_0 is not rejected,
- the data sample D_1 is “not compatible” with H_0 with significance level α : the hypothesis H_0 is rejected in favour of the alternative hypothesis

$$H_1 : (\varepsilon, \rho, \tilde{v}_0) \neq (\varepsilon^*, \rho^*, \tilde{v}_0^*). \quad (6)$$

Assigned α the decision rule defines what to do to evaluate the compatibility of the data sample D_1 with the null hypothesis H_0 .

3.3.7 The Test Statistic

In Fatone et al. Journal of Inverse and Ill Posed Problems 2013 starting from the data sample $\hat{\xi}_i, i = 1, 2, \dots, M$, we compute:

$$\hat{F}_k(M) = \frac{1}{M} \sum_{i=1}^M w_i \hat{\xi}_i^k, \quad k = 2, 3, 4, \quad (7)$$

where $w_i, i = 1, 2, \dots, M$, are positive weights decreasing when i increases.

Remark

The quantity \hat{F}_k is a realization of the random variable:

$$F_k(M) = \hat{f}_k(\eta_{t_1}, \eta_{t_2}, \dots, \eta_{t_M}), \quad k = 2, 3, 4, \quad (8)$$

where

$$\hat{f}_k(\eta_{t_1}, \eta_{t_2}, \dots, \eta_{t_M}) = \frac{1}{M} \sum_{i=1}^M w_i \eta_{t_i}^k, \quad k = 2, 3, 4, \quad (9)$$

and $\eta_{t_i} = \xi_{t_i} | \{ \xi_{t_k} = \hat{\xi}_k, k = 0, 1, \dots, i - 1 \}$, that is η_{t_i} is the random variable ξ_{t_i} conditioned to $\xi_{t_k} = \hat{\xi}_k, k = 0, 1, \dots, i - 1, i = 1, 2, \dots, M$. The random variables F_2, F_3, F_4 are used to build the vector valued test statistic $F = (F_2, F_3, F_4)$ of the statistical test.

Note that the random variables $F_k(M), k = 2, 3, 4$, in (8) are unbiased estimators of:

$$\begin{aligned} \mathcal{F}_k^*(M) &= \int_{-\infty}^{+\infty} d\xi_1 \cdots \int_{-\infty}^{+\infty} d\xi_M f_k(\xi_1, \xi_2, \dots, \xi_M) \\ \hat{p}_N(\xi_1, t_1, \xi_2, t_2, \dots, \xi_M, t_M | \tilde{\xi}_0, \tilde{v}_0, t_0), k &= 2, 3, 4, \end{aligned} \quad (10)$$

where $\hat{p}_N(\xi_1, t_1, \xi_2, t_2, \dots, \xi_M, t_M | \tilde{\xi}_0, \tilde{v}_0, t_0)$ is the joint probability density function associated to the normal SABR model (1)-(4) of having $\xi_{t_i} = \xi_i, i = 1, 2, \dots, M$, conditioned to $\xi_{t_0} = \tilde{\xi}_0$ and $v_{t_0} = \tilde{v}_0$. Recall that $t_0 = 0$.

In (10) we have chosen:

$$\begin{aligned} f_k(\xi_1, \xi_2, \dots, \xi_M) &= \frac{1}{M} \left[w_1 \xi_1^k + \sum_{i=2}^M w_i \xi_i^k \prod_{j=1}^{i-1} \delta(\xi_j - a_j) \right], \\ k &= 2, 3, 4, \end{aligned} \quad (11)$$

where w_i are positive weights decreasing when i increases and $a_i = \hat{\xi}_i$, $i = 1, 2, \dots, M$.

The joint probability density function in (10) is given by:

$$\begin{aligned} \hat{p}_N(\xi_1, t_1, \xi_2, t_2, \dots, \xi_M, t_M | \tilde{\xi}_0, \tilde{v}_0, t_0) &= \int_0^{+\infty} dv_1 \\ &\int_0^{+\infty} dv_2 \cdots \int_0^{+\infty} dv_M p_N(\xi_1, v_1, t_1, \tilde{\xi}_0, \tilde{v}_0, t_0) \cdot \\ &p_N(\xi_2, v_2, t_2, \xi_1, v_1, t_1) \cdots \\ &p_N(\xi_M, v_M, t_M, \xi_{M-1}, v_{M-1}, t_{M-1}), \end{aligned} \quad (12)$$

where $p_N(\xi_{i+1}, v_{i+1}, t_{i+1}, \xi_i, v_i, t_i)$ is the transition probability density function associated to the normal SABR model (1)-(4) of having $\xi_{t_{i+1}} = \xi_{i+1}$, $v_{t_{i+1}} = v_{i+1}$ given the fact that $\xi_{t_i} = \xi_i$, $v_{t_i} = v_i$, $i = 0, 1, \dots, N - 1$.

In Fatone et al. Journal of Mathematical Finance 2013 we have derived a formula that gives the transition probability density function of the normal SABR model p_N as a one dimensional integral of an explicitly known integrand.

3.3.8 Monte Carlo Method

When M is “large” (i.e. M is greater than 3 or 4) the integrals (10) and (12) are high dimensional integrals that must be evaluated using Monte Carlo method.

In the straightforward application of the Monte Carlo method to the evaluation of $\mathcal{F}_k^*(M)$ we must draw a sample from the probability density

functions $p_N(\xi_{i+1}, v_{i+1}, t_{i+1}, \xi_i, v_i, t_i)$, $i = 0, 1, \dots, M - 1$. However the complexity of the expression of p_N makes difficult to draw this sample.

This difficulty can be overcome using the importance sampling method, that allows to draw the sample of the Monte Carlo procedure from probability density functions that are similar to the density functions $p_N(\xi_{i+1}, v_{i+1}, t_{i+1}, \xi_i, v_i, t_i)$, $i = 0, 1, \dots, M - 1$, and that are easy to sample. These new probability density functions are called *sampling distributions*.

3.3.9 Importance Sampling

The sampling distribution used to evaluate (10), (12) are obtained substituting to the model (1)-(4) the following simplified model:

$$d\xi_t = \tilde{v}_0^* dB_t^1, \quad t > 0, \tag{13}$$

$$dv_t = \varepsilon^* v_t dB_t^2, \quad t > 0, \tag{14}$$

where the stochastic processes $B_t^1, B_t^2, t > 0$, in (13), (14) are uncorrelated standard Wiener processes such that $B_0^1 = B_0^2 = 0$, and $dB_t^1, dB_t^2, t > 0$, are their stochastic differentials. Equations (13), (14) are equipped with the initial conditions: $\xi_0 = \tilde{\xi}_0$ and $v_0 = \tilde{v}_0^*$.

For $i = 0, 1, \dots, M - 1$ the “sampling distribution” (density function) used to approximate $p_N(\xi_{i+1}, v_{i+1}, t_{i+1}, \xi_i, v_i, t_i)$ is given by:

$$\frac{1}{\sqrt{2\pi(t_{i+1} - t_i)\tilde{v}_0^2}} \exp \left[-\frac{1}{2(t_{i+1} - t_i)\tilde{v}_0^{*2}} (\xi_{i+1} - \xi_i)^2 \right] \cdot \frac{1}{\sqrt{2\pi(t_{i+1} - t_i)\varepsilon^{*2}}} \exp \left[-\frac{1}{2(t_{i+1} - t_i)\varepsilon^{*2}} (\log(v_{i+1}) - \log(v_i) + \frac{1}{2}\varepsilon^{*2}(t_{i+1} - t_i))^2 \right]. \tag{15}$$

The function (15) is the probability density function associated to the simplified model (13), (14).

3.3.10 Ad hoc Statistical Test (more)

The statistical test Fatone et al. Journal of Inverse and Ill Posed Problems 2013 is based on the idea of comparing the test statistic vector $F = (F_2, F_3, F_4)$ observed in the data sample D_1 (i.e. its realization $\hat{F} = (\hat{F}_2, \hat{F}_3, \hat{F}_4)$) with the vector $\hat{\mathcal{F}}^* = (\hat{\mathcal{F}}_2^*, \hat{\mathcal{F}}_3^*, \hat{\mathcal{F}}_4^*)$. That is we evaluate $\|\hat{F} - \hat{\mathcal{F}}^*\|$, where $\|\cdot\|$ is the norm of \cdot in \mathbb{R}^3 .

The heuristic decision rule of the test is:

- do not reject H_0 if $\|\hat{F} - \hat{\mathcal{F}}^*\|$ is “small”;
- reject H_0 if $\|\hat{F} - \hat{\mathcal{F}}^*\|$ is “big”.

In order to give a precise quantitative meaning to the expressions “big” and “small” of the heuristic decision rule:

- we must know the distribution of the random variable $F = (F_2, F_3, F_4)$. The distribution of F can be approximated using a sample of values of F generated numerically (i.e. statistical simulation),
- we must determine a threshold value that divides the space of the possible values assumed by F into two regions: the acceptance region and the rejection region. When the value \hat{F} belongs to the acceptance region, the null hypothesis is accepted, or at any rate not rejected, otherwise the null hypothesis is rejected,

- the threshold value that determines the acceptance and the rejection region depends on the significance level α .

3.3.11 Sampling Distribution

We approximate the joint probability density function of the test statistic $F = (F_2, F_3, F_4)$ with a three-dimensional histogram obtained from a sample of the random vector $F = (F_2, F_3, F_4)$ obtained integrating numerically the normal SABR model when H_0 is true with the explicit Euler method.

Analogously the integral of the joint probability density function of F_2, F_3, F_4 on a subset of \mathbb{R}^3 is approximated with the appropriate sum of relative frequencies of sampled points computed in the three-dimensional histogram.

3.3.12 Threshold Value

Let us assign a significance level $\alpha \in (0, 1)$. Recall that α is the maximum probability of making a type I error (i.e. rejecting H_0 when H_0 is true). Let us denote the threshold value with $s_{\alpha, M}$. We choose $s_{\alpha, M}$ as the infimum of the values $A_{\alpha, M}$ such that:

$$Probability(\|F - \hat{\mathcal{F}}^*\| \geq A_{\alpha, M}) \leq \alpha, \quad (16)$$

where the $Probability(\|F - \hat{\mathcal{F}}^*\| \geq A_{\alpha, M})$ is determined integrating the joint probability density function of F_2, F_3, F_4 (when H_0 is true) outside the sphere of center $\hat{\mathcal{F}}^*$ and radius $A_{\alpha, M} > 0$, i.e. summing the relative frequencies of the sampled points outside the sphere of the three-dimensional histogram that approximates the joint probability density function of F_2, F_3, F_4 .

3.3.13 Decision Rule

Summarizing, given the significance level $\alpha \in (0, 1)$, the sample size M , the null hypothesis H_0 and the corresponding threshold value $s_{\alpha, M}$ the decision rule of the statistical test is given by:

- if $\|\hat{F} - \hat{\mathcal{F}}^*\| \leq s_{\alpha, M}$ do not reject H_0 with significance level α ;
- if $\|\hat{F} - \hat{\mathcal{F}}^*\| > s_{\alpha, M}$ reject H_0 with significance level α .

3.3.14 An Example Using Synthetic Data

Let us choose:

$$(\varepsilon, \rho, \tilde{v}_0) = (0.1, -0.2, 0.5)$$

as the unknown parameters of the normal SABR model that we want recover in the calibration problem. For $i = 1, 2, \dots, M$ let us choose the weights w_i , and the constants a_i appearing in (11) as follows: $w_i = \exp(-2 * (i - M))$, $a_i = \hat{\xi}_i$.

- **Data sample generation:** Let $M = 10$ be the number of observations. Let $\Delta t = 20$ be a time increment and $t_i = i\Delta t$, $i = 0, 1, \dots, M$, be a discrete set of observation times. The synthetic data $\hat{\xi}_i$, $i = 1, 2, \dots, M$, are obtained approximating with the explicit Euler method (1)-(4) one trajectory of the normal SABR model with the previous choice of the parameter values. That is: $D_1 = \{\hat{\xi}_i, i = 1, 2, \dots, M\}$ is the data sample of the numerical example presented. Using D_1 we compute $\hat{F} = (1.684, 6.435, 24.604)$.
- **Hypothesis formulation:** We proceed as suggested in Fatone et al. Journal of Inverse and Ill Posed Problems 2013 and we solve the

calibration problem using the nonlinear least squares method. The solution obtained with the least squares method is used to formulate the null hypothesis of the statistical test. That is we consider the following null hypothesis:

$$H_0 : (\varepsilon, \rho, \tilde{v}_0) = (0.1261, -0.3356, 0.515). \quad (17)$$

We test the hypothesis H_0 with statistical significance α using the sample data D_1 .

- Monte Carlo method: We compute $\mathcal{F}^*(M) = (\mathcal{F}_1^*(M), \mathcal{F}_2^*(M), \mathcal{F}_3^*(M))$, where $\mathcal{F}_k^*(M)$, $k = 2, 3, 4$, are given by (8), when H_0 is true, using the Monte Carlo procedure. We have $\mathcal{F}^*(M) = (0.607, 0.117, 16.449)$.
- Threshold value determination: We build a sample of $N = 1000$ realizations of the random variables $F_2(M)$, $F_3(M)$, $F_4(M)$ defined in (8) when H_0 is true integrating numerically with explicit Euler method the normal SABR model (1)-(4) on the time interval $[0, 200]$. The values of $s_{\alpha,10}$ as a function of α obtained in this way are shown in the following table:

$H_0 : (\varepsilon, \rho, \tilde{v}_0) = (0.1261, -0.3356, 0.515), \quad M = 10$	
α	$s_{\alpha,10}$
0.01	571
0.05	179
0.1	50

- Decision rule:

- if $\|\hat{F} - \hat{F}^*\| \leq s_{\alpha,10}$ do not reject H_0 , with significance level α ;
- if $\|\hat{F} - \hat{F}^*\| > s_{\alpha,10}$ reject H_0 , with significance level α .

The hypothesis H_0 is retained for the values of α considered in the Table.

3.3.15 References

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