Chapter 10

Post-Newtonian Approximation in the Universe

In chapter V the post-Newtonian approximation neglecting the universe, i.e. in empty space is given. In this chapter post-Newtonian approximation in the universe is studied. Here, we follow along the lines of article [Pet 00].

10.1 Post-Newtonian Approximation

The metric is the pseudo-Euclidean geometry given by equation (1.1) with (1.5). The potentials are:

$$\begin{aligned} g_{ij} &= a^2\left(1+\frac{2}{c^2}U\right)\delta_{ij},\ (i;j=1,2,3) \\ &= -\frac{4}{c^2}\frac{a}{\sqrt{h}}V_i,\ (i=1,2,3;\ j=4) \\ &= -\frac{4}{c^2}\frac{a}{\sqrt{h}}V_j,\ (i=4;\ j=1,2,3) \\ &= -\frac{1}{h}\left(1-\frac{2}{c^2}U+\frac{1}{c^4}S\right).(i=j=4) \end{aligned} \tag{10.1}$$

The matter tensor is given by (5.1a) without the factor $\left(\frac{-G}{-\eta}\right)^{1/2}$ in contrast to the considerations of chapter V. In the following, we follow along the lines of chapter V. Let us assume condition (5.5) and (5.6b). Again we get (5.6d). Relation (1.12) with (1.13) implies by the use of (10.1)

$$\frac{dt}{d\tau} = 1 + \frac{1}{c^2}U + \frac{1}{2c^2}a^2h|v|^2. \tag{10.2}$$

The matter tensor has to post-Newtonian accuracy the form

$$\begin{aligned} T(M)^i_j &= a^2\rho h v^i v^j + pc^2\delta_{ij},\ (i;j=1,2,3) \\ &= a^2\rho h c v^i\left(1+\frac{\Pi}{c^2}+\frac{p}{\rho}+\frac{4}{c^2}U+\frac{1}{c^2}a^2h|v|^2\right) \\ &\quad -\frac{4}{c}a\sqrt{h}\rho v^i,\ (i=1,2,3;\ j=4) \\ &= -\rho c v^j\left(1+\frac{\Pi}{c^2}+\frac{p}{\rho}+\frac{1}{c^2}a^2h|v|^2\right),(i=4;\ j=1,2,3) \\ &= -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{1}{c^2}a^2h|v|^2\right).(i=j=4). \end{aligned} \tag{10.3}$$

Now, we can receive U, V_i and S to post-Newtonian approximation in the homogeneous, isotropic universe similar to chapter 5.1. It follows

$$U = k\frac{\sqrt{h}}{a}\int \frac{\rho'}{|x-x'|} d^3x'. \tag{10.4}$$

Here, the integral is taken over the whole space and $\rho' = \rho(x', t)$, etc. We introduce

$$\chi = -k\frac{\sqrt{h}}{a}\int \rho' |x - x'|\, d^3x'. \tag{10.5}$$

We get

$$V_i = kh\int \rho' \frac{v^i{}'}{|x-x'|} d^3x' + \frac{1}{4}k\frac{dh}{dt}\int \rho' \frac{x^i - x^i{}'}{|x-x'|} d^3x'. \tag{10.6}$$

We introduce the potential

$$\psi = -kh\int \rho' \frac{(v', x-x')}{|x-x'|} d^3x' + \frac{1}{2}\frac{a}{\sqrt{h}}\frac{dh}{dt}\chi \tag{10.7a}$$

where $(v', x - x')$ denotes the scalar- product in R^3. It follows

$$S = 2U^2 + 2\frac{a}{\sqrt{h}}\psi + \left[16\pi k\left(\frac{\rho_{m0}}{a} + 2\frac{\rho_{r0}}{a^2}\right) - 2\Lambda c^2 a^2\right]\chi$$
$$+ a^2 h\left\{\frac{d^2\chi}{dt^2} + \left(3\frac{a'}{a} + \frac{1}{2}\frac{h'}{h}\right)\frac{d\chi}{dt}\right\} - 4ah^{3/2}\phi_1 - 2\frac{\sqrt{h}}{a}\phi_3 - 6\frac{\sqrt{h}}{a}\phi_4 \tag{10.7b}$$

where ϕ_1, ϕ_3 and ϕ_4 are given by (5.14).

10.2 Equations of Motion

The conservation law of mass (1.27) implies by the use of

$$\rho^* = \rho\frac{dt}{d\tau} = \rho\sqrt{h}\left(1 + \frac{1}{c^2}U + \frac{1}{2c^2}a^2 h|v|^2\right) \tag{10.8}$$

to $O\left(\frac{1}{c^2}\right)$ the well-known conservation law

$$\frac{\partial \rho^*}{\partial t} + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\rho^* v^k\right) = 0. \tag{10.9}$$

Hence, the conserved mass m is

$$m = \int \rho^*(x', t)\, d^3x'. \tag{10.10}$$

The equations of motion (5.35) use the Christoffel symbols $\Gamma(g)^{i}_{jk}$ which are omitted and can be found in the appendix of [Pet 00]. It is worth to mention that we have no gauge problem in contrast to the theory of general relativity considered by Shibata et al. [Shi 95] which implies some difficulties. To get the equations of motion given by (5.35) to post-Newtonian accuracy the energy tensor of matter $T(M)^{44}$ must be calculated to $O\left(\frac{1}{c^4}\right)$. Let us introduce the velocity

$$\bar{v}^{i} = a^{2}\sqrt{h}v^{i}\left(1+\frac{1}{c^{2}}\left(\Pi+\frac{pc^{2}}{\rho}+U+\frac{1}{2}a^{2}h|v|^{2}\right)\right). \tag{10.11}$$

Replace U of (10.4) by

$$U^{*} = k\int\frac{\rho^{*\prime}}{|x-x'|}d^{3}x'. \tag{10.12}$$

Put

$$A(t)=\frac{a}{\sqrt{h}}\left\{h\left(-3\left(\frac{a'}{a}\right)^{2}+\frac{a'}{a}\frac{h'}{h}+\frac{3}{4}\left(\frac{h'}{h}\right)^{2}\right)+H_{0}^{2}\left(\frac{21}{4}\frac{\Omega_{m}}{a^{3}}+\frac{23}{2}\frac{\Omega_{r}}{a^{4}}-\frac{21}{2}\Omega_{\Lambda}\right)\right\} \tag{10.13}$$

Then, the equations of motion to $O\left(\frac{1}{c^2}\right)$ are (i=1,2,3):

$$\begin{aligned}
&\frac{\partial\bar{v}^{i}}{\partial t}+\sum_{k=1}^{3}v^{k}\frac{\partial\bar{v}^{i}}{\partial x^{k}}\\
&=-\frac{1}{\rho^{*}}\frac{\partial pc^{2}}{\partial x^{i}}+\frac{1}{a\sqrt{h}}\frac{\partial U^{*}}{\partial x^{i}}+\frac{1}{c^{2}}\frac{2}{a}U^{*}\frac{1}{\rho^{*}}\frac{\partial pc^{2}}{\partial x^{i}}\\
&\quad-\frac{7}{2}\frac{1}{c^{2}}\frac{a'}{a}\frac{k}{a}\int\rho^{*\prime}\frac{\bar{v}^{i\prime}}{|x-x'|}d^{3}x'\\
&\quad-\frac{1}{2}\frac{1}{c^{2}}\frac{k}{a}\int\rho^{*\prime}\frac{(\bar{v}',x-x')(x^{i}-x^{i\prime})}{|x-x'|^{3}}d^{3}x'\\
&\quad+\frac{1}{c^{2}}A(t)k\int\rho^{*\prime}\frac{x^{i}-x^{i\prime}}{|x-x'|}d^{3}x'\\
&\quad+\frac{1}{c^{2}}\frac{1}{a\sqrt{h}}\left(\Pi+3\sqrt{h}\frac{pc^{2}}{\rho^{*}}+\frac{3}{2}\frac{|\bar{v}|^{2}}{a^{2}}-\frac{3}{a}U^{*}\right)\frac{\partial U^{*}}{\partial x^{i}}\\
&\quad-\frac{1}{c^{2}}\frac{k}{a\sqrt{h}}\int\rho^{*\prime}\Pi'\frac{x^{i}-x^{i\prime}}{|x-x'|^{3}}d^{3}x'
\end{aligned} \tag{10.14}$$

$$-\frac{2}{c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}|\bar{v}'|^2\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{1}{c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}U^{*\prime}\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{3}{c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}\frac{(\bar{v}',x-x')\bar{v}^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{3}{2c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}\frac{(\tilde{v}',x-x')^2(x^i-x^{i\prime})}{|x-x'|^3}d^3x'$$

$$+\frac{7}{2c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}\frac{\partial U^{*\prime}}{\partial x^{i\prime}}\frac{1}{|x-x'|}d^3x'$$

$$+\frac{1}{2c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}\left(\sum_{k=1}^{3}\left(x^k-x^{k\prime}\right)\frac{\partial U^{*\prime}}{\partial x^{k\prime}}\right)\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$-\frac{4}{c^2}\frac{k}{a}\int \rho^{*\prime}\left\{v^{i\prime}\frac{(v',x-x')}{|x-x'|^3}-\left(x^i-x^{i\prime}\right)\frac{|v'|^2}{|x-x'|^3}\right\}d^3x'$$

$$-\frac{2}{c^2}\frac{k}{a}v^i\int \rho^{*\prime}\frac{(v',x-x')}{|x-x'|^3}d^3x'-\frac{2}{c^2}a\sqrt{h}v^i\left(\sum_{k=1}^{3}v^k\frac{\partial U^*}{\partial x^k}-\frac{a'}{a}U^*\right).$$

In the article [Shi 95] the special case $p=\Pi=0$ is considered. Furthermore, the two results of [Shi 95] and the equations of motion (10.14) with (10.8) cannot be compared with one another because the time-derivatives on the right side in [Shi 95] are not completely eliminated. In addition, the function $h(t)$ does not appear in Einstein's cosmological models. By the introduction of the proper time $\tilde{\tau}$ given by (8.1) the function $h(t)$ can be eliminated by

$$\tilde{v}^i=\frac{dv^i}{d\tilde{\tau}}=v^i\frac{dt}{d\tilde{\tau}}=v^i\sqrt{h},\ \frac{da}{d\tilde{\tau}}=\frac{da}{dt}\frac{dt}{d\tilde{\tau}}=a'\sqrt{h}. \tag{10.15}$$

By multiplication of relation (10.14) with $\sqrt{h}$ and the use of (10.15) it follows that h does not appear in the new equations of motion by the use of the proper time $\tilde{\tau}$ which is used by general relativity.

It is worth to mention that the special case where the universe is neglected, i.e.

$$\Omega_m=\Omega_r=\Omega_\Lambda=0, a(t)=h(t)=1,$$

the equations of motion (10.14) are studied in chapter 5.1 where the explicit form of the equations is not stated. In chapter V the energy-momentum (5.1) is used to compare post-Newtonian approximation of flat space-time of gravitation and of general relativity.

All the calculations of the represented results in this chapter can be found in the article [Pet 00].

10.3 Newtonian and Long-Field Forces

In this sub-chapter we will consider only the post-Newtonian long-field cosmological expression

$$F_i^U = \frac{1}{c^2} A(t) k \int \rho^{*\prime} \frac{x^i - x^{i\prime}}{|x - x\prime|} d^3x' \tag{10.16}$$

of the non-stationary universe and compare (10.16) with the Newtonian force

$$F_i^N = -\frac{1}{a\sqrt{h}} k \int \rho^{*\prime} \frac{x^i - x^{i\prime}}{|x - x'|^3} d^3x'. \tag{10.17}$$

In the following we consider spherical symmetry with Euclidean distance r from the centre of the body. We get

$$F_i^U = A(t) \frac{4\pi k}{c^2} \left(\int_0^r \rho^*(r') r'^2 \left(1 - \frac{1}{3} \left(\frac{r\prime}{r} \right)^2 \right) dr' + \frac{2}{3} r \int_r^\infty \rho^*(r') r' \, dr' \right) \frac{x^i}{r}$$

and

$$F_i^N = -\frac{1}{a\sqrt{h}} \frac{4\pi k}{r^2} \int_0^r \rho^* (r') r'^2 dr' \frac{x^i}{r}.$$

Let us assume that the radius r is greater than that of the distribution of matter and the post-Newtonian force expression (10.16) compensates the Newtonian force. We get compensation of the two forces for

$$\frac{1}{a\sqrt{h}} \frac{1}{r^2} \approx \frac{A(t)}{c^2}. \tag{10.18}$$

Hence, it follows

$$r \approx c / \left(a\sqrt{h} A(t) \right)^{1/2}. \tag{10.19}$$

Equation (9.3) implies that at present time $t = 0$, i.e. $a(0) = h(0) = 1$,and $A(0) \approx H_0^2$. Therefore, the post-Newtonian force $F_i^U$is only important on very large scales.

Let us consider the universe at later times, i.e. in the future. Then, it holds with

$$H_0\tilde{t}_1 \approx -1/\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)$$

for $H_0 t \gg H_0\tilde{t}_1$ by the results of chapter VII the relations

$$a^3(t) \approx \left(\frac{3}{2}\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0 t - H_0\tilde{t}_1)^2 \Big/ \left(3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}\right),$$

$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}.$$

Hence, for

$$H_0 t - H_0\tilde{t}_1 \gg \left(1-\sqrt{\Omega_m}\right)^2 \big/ \left(3\Omega_m\sqrt{\Omega_\Lambda}\right)$$

we have

$$a^3(t) \approx \frac{3}{4}\frac{1}{\sqrt{\Omega_\Lambda}}\left(\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0 t - H_0\tilde{t}_1),$$
$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1). \tag{10.20}$$

Elementary calculations yield

$$a\sqrt{h}A(t) \approx \frac{39}{2}\Omega_\Lambda H_0^2 a^2. \tag{10.21}$$

Hence, we get by the use of (10.19)

$$r \approx \sqrt{\frac{2}{39\Omega_\Lambda}}\left(\frac{4\sqrt{\Omega_\Lambda}\left(1-\sqrt{\Omega_\Lambda}\right)^2}{3{\Omega_m}^2}\right)^{1/3} \frac{c}{H_0}\frac{1}{(H_0 t - H_0\tilde{t}_1)^{1/3}}. \tag{10.22}$$

Therefore, the radius where the two forces compensate one another is decreasing with increasing time under the assumption $\Omega_\Lambda > 0$.

The case $\Omega_\Lambda = 0$ gives under the assumption (7.44b) by virtue of (7.45b) and (7.46) the solutions for the universe

$$a^3(t) \approx \frac{4}{9}\frac{1}{\Omega_m}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 (H_0 t - H_0\tilde{t}_1)^2,$$
$$\sqrt{h(t)} \approx 1 - \frac{1}{9}K_0, \quad \Omega_m \approx 1. \tag{10.23}$$

Elementary calculations imply

$$a\sqrt{h}A(t) \approx \frac{9}{4}H_0^2\frac{1}{a}.$$

This result yields by the use of (10.19)

$$r \approx \frac{2}{3}\frac{c}{H_0}\sqrt{a(t)} \approx C_1\frac{c}{H_0}(H_0 t - H_0\tilde{t}_1)^{1/3} \tag{10.24}$$

with a suitable constant C_1. Hence,the radius where the two forces compensate one another is increasing in the course of time.

Therefore, the radius where two forces compensate one another are quite different for the two cases. This radius decreases in a universe with $\Lambda > 0$ and increases in a universe with $\Lambda = 0$ in the course of time.

Summarizing, it follows that in the neighbourhood of a spherically symmetric body the large scale-structure in the universe is not important compared to the Newton force of this body. These results are also contained in the article [Pet 00].