Chapter 6

Zweier I-Convergent Sequence Spaces Defined by a Sequence of Moduli

"The essence of mathematics lies in its freedom"- Cantor.

6.1 Introduction

Recently Khan and Ebadullah[31] introduced the following classes of sequences

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L}\in \mathbb{C} \ \} \in I\},$$

$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f).$$

In this chapter we introduce the following class of sequence spaces.

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some L}\in \mathbb{C} \ \} \in I\},$$

$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty(F) \cap \mathcal{Z}^I(F)$$

and

$$m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty(F) \cap \mathcal{Z}_0^I(F).$$

6.2 Main Results

Theorem 6.2.1. For a sequence of modulii $F = (f_k)$, the classes of sequences $\mathcal{Z}^I(F), \mathcal{Z}_0^I(F), m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C} \quad ;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C} \quad ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \qquad [6.1]$$

$$A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \qquad [6.2]$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|)$$
$$\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|)$$

Now, by [6.1] and [6.2], $\{k \in \mathbb{N}$: $f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\}$ $\subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$. Hence $\mathcal{Z}^I(F)$ is a linear space.

We state the following result without proof in view of Theorem 6.2.1.

Theorem 6.2.2. The spaces $m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are normed linear spaces, normed by

$$||x_k||_* = \sup_k f_k(|x_k|). \qquad [6.3]$$

Theorem 6.2.3. A sequence $x = (x_k) \in m_{\mathcal{Z}}^I(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathcal{Z}}^I(F). \qquad [6.4]$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m_{\mathcal{Z}}^I(F). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$.Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathcal{Z}}^I(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathcal{Z}}^I(F)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathcal{Z}}^I(F)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m_{\mathcal{Z}}^I(F) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m_{\mathcal{Z}}^I(F)$ as well as $C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I(F)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I(F)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m_{\mathcal{Z}}^I(F)$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m_{\mathcal{Z}}^I(F)$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$.

Theorem 6.2.4. Let (f_k) and (g_k) be modulus functions for some fixed k that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ etc, then the following assertions hold.
(a) $X(g_k) \subseteq X(f_k.g_k)$,
(b) $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

Proof. (a) Let $(x_n) \in \mathcal{Z}_0^I(g_k)$. Then

$$I - \lim_n g_k(|x_n|) = 0 \qquad [6.5]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 < t < \delta$. Write $y_n = g_k(|x_n|)$ and consider $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n<\delta} + \lim_n f_k(y_n)_{y_n>\delta}$. We have

$$\lim_n f_k(y_n) \leq f_k(2) \lim_n (y_n). \qquad [6.6]$$

For $y_n > \delta$, we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since f_k is non-decreasing, it follows that

$$f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2}f_k(2) + \frac{1}{2}f_k(\frac{2y_n}{\delta})$$

Since f_k satisfies the $\triangle_2$-condition, we have

$$f_k(y_n) < \frac{1}{2}K\frac{y_n}{\delta}f_k(2) + \frac{1}{2}K\frac{y_n}{\delta}f_k(2) = K\frac{y_n}{\delta}f_k(2)$$

Hence

$$\lim_n f_k(y_n) \leq \text{max(1,K)}\delta^{-1} f_k(2) \lim_n (y_n). \qquad [6.7]$$

From [6.5], [6.6] and [6.7], we have $(x_n) \in \mathcal{Z}_0^I(f_k.g_k)$.

Thus $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k.g_k)$. The other cases can be proved similarly.

(b) Let $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$. Then

$$I - \lim_n f_k(|x_n|) = 0 \; and \; I - \lim_n g_k(|x_n|) = 0$$

The rest of the proof follows from the following equality

$$\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|).$$

Corollary 6.2.5. $X \subseteq X(f_k)$ for some fixed k and X =$\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 6.2.6. The spaces $\mathcal{Z}_0^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(F)$. Let $(x_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim_k f_k(|x_k|) = 0 \qquad [6.8]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [6.8] and the following inequality

$$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|) \; for \; all \; k \in \mathbb{N}.$$

That the space $\mathcal{Z}_0^I(F)$ is monotone follows from the Lemma 6.1.1. For $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly.

Theorem 6.2.7. The spaces $\mathcal{Z}^I(F)$ and $m_{\mathcal{Z}}^I(F)$ are neither solid nor monotone in general .

Proof. Here we give a counter example. Let $I = I_\delta$ and $f_k(x) = x^2$ for some fixed k and for all $x \in [0, \infty)$. Consider the K-step space $X_K(f_k)$ of X defined as follows.

Let $(x_n) \in X$ and let $(y_n) \in X_K$ be such that

$$(y_n) = \begin{cases} (x_n), & \text{if n is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence (x_n) defined by $(x_n) = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(F)$. Thus $\mathcal{Z}^I(F)$ is not monotone. Hence $\mathcal{Z}^I(F)$ is not solid.

Theorem 6.2.8. The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(F)$ is a sequence algebra. Let
$(x_k), (y_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0$$

Then we have

$$I - \lim f_k(|(x_k.y_k)|) = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I(F)$ is a sequence algebra. For the space $\mathcal{Z}^I(F)$, the result can be proved similarly.

Theorem 6.2.9. The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $f_k(x) = x^3$ for some fixed k and for all $x \in [0, \infty)$. Consider the sequence (x_n) and (y_n) defined by

$$x_n = \frac{1}{n} \quad and \;\; y_n = n \;\text{ for all } n \in \mathbb{N}$$

Then $(x_n) \in \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$, but $(y_n) \notin \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$. Hence the spaces $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free.

Theorem 6.2.10. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f_k(x) = x$ for some fixed k and for all $x \in [0, \infty)$.

If

$$x_n = \begin{cases} 1, & \text{for } n \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.22 $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(n) = \begin{cases} \phi(n), & \text{for } n \in K, \\ \psi(n), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(n)} \notin \mathcal{Z}^I(F)$ and $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$. Hence $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.

Theorem 6.2.11. $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f_k(|x_k - L|) = 0$$

We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I(F)$. The inclusion $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ is obvious.

Theorem 6.2.12. The function $\hbar : m_\mathcal{Z}^I(F) \to \mathbb{R}$ is the Lipschitz function, where $m_\mathcal{Z}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x-y||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x-y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x-y||_*\} \in m^I_{\mathcal{Z}}(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x-y||_*\} \in m^I_{\mathcal{Z}}(F).$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}(F)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x-y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly.

Theorem 6.2.13. If $x, y \in m^I_{\mathcal{Z}}(F)$, then $(x.y) \in m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I_{\mathcal{Z}}(F),$$

$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I_{\mathcal{Z}}(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \qquad [6.9] \end{aligned}$$

As $m^I_{\mathcal{Z}}(F) \subseteq \mathcal{Z}^I_\infty(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn[6.9] we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly.