Chapter 3

Method of Moments in the Theory of Singularly Perturbed Systems

After applying the method of moments to solve the problem of optimal control of linear systems with lumped parameters by Krasovsky N. N. [86] this method has been successfully compiled and turned out to be a very strong unit study many tasks of control. In this chapter will set out how to use the theory of moments to the problem of control of singularly perturbed systems with a variety of optimizable functionals.

3.1 Statement of the Problem on How to Manage the Problem of Moments

Let the behavior of the controlled system described by the equations

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + B_1(t)u + f_1(t), \quad x(t_0) = x^0, \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + B_2(t)u + f_2(t), \quad z(t_0) = z^0, \end{aligned} \tag{3.1.1}$$

where $x(t) \in R^n$, $z(t) \in R^m$ - vectors of state, $u(t) \in R^r$ - control, $f_1(t) \in R^n$, $f_2(t) \in R^m$ - constantly operating outside forces; $t \in [t_0, t_1]$, μ - "small" positive parameter $(0 < \mu << 1)$.

It is assumed that the system

$$\dot{\overline{x}} = A_0(t)\overline{x} + B_0(t)u + f_0(t) \tag{3.1.2}$$

where $A_0(t) = A_1(t) - A_2(t) \cdot A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t) \cdot A_4^{-1}(t)B_2(t),$

$$f_0(t) = f_1(t) - A_2(t) \cdot A_4^{-1}(t) f_2(t)$$

is completely controllable and

$$\operatorname{Re} \lambda(A_4(t)) < 0\,. \tag{3.1.3}$$

It should be noted that some problems of control chosen value which characterizes the costs of resources for the implementation of the process control. Usually is required to achieve the desired result so that the value of this quantity was minimal and this value does not exceed certain limits. This value is called the criterion of optimality or intensity [86] control and denote it by the symbol $J(u)$.

Let any normed space of functions by symbol $M\{\cdot\}$ and will use it whenever the norm of the space of functions is not fixed. The symbol R_μ we will denote the perturbed problem, a R_0 - unperturbed problem of optimal control and at the same time J_μ^*, J_0^* – the minimum values of the criterion of optimality in problems R_μ^* and R_0 respectively.

We formulate the following problem R_μ : Let was selected criterion of optimality $J_\mu(u_\mu)$, which can be interpreted as a norm $\rho_\mu^*(u_\mu)$ functions $u_\mu(t) = u(t,\mu)$ in space $M^*\{u_\mu\}$.

It requires among admissible controls [86] to find the optimal control $u_\mu^0(t)$, which puts the system (3.1.1) from the initial state $x(t_0,\mu) = x^0$, $z(t_0,\mu) = z^0$ to the final state $x(t_1,\mu) = x^1$, $z(t_1,\mu) = z^1$ and thus having the smallest possible form $\rho_\mu(u_\mu^0)$.

This problem with the $\mu = 0$ responsible task of smaller dimension R_0 :

$$J_0(\overline{u}) \to \min_{\overline{u} \in M^0},$$

$$\overline{x} = A_0(t)\overline{x} + B_0(t)\overline{u} + f_0(t), \overline{x}(t_0) = x^0,$$

$$\overline{z} = -A_4^{-1}(t)[A_3(t)\overline{x} + B_2(t)\overline{u} + f_2(t)],$$

where $f_0(t) = f_1(t) - A_2(t) \cdot A_4^{-1}(t) f_2(t)$.

As shown in Chapter 1, the system (3.1.1) may be replaced system with separate movements

$$\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu),\ \tilde{x}(t_0) = \tilde{x},$$
$$\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu),\ \tilde{z}(t_0) = \tilde{z}^0, \tag{3.1.4}$$

where

$$\tilde{A}_1(t) = \tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu),\ \tilde{A}_4(t) = \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$$

$$\tilde{B}_1(t) = \tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,\mu),\ \tilde{B}_2(t) = \tilde{B}_2(t,\mu) = B_2(t) - \mu H(t,\mu)B_1(t), \tag{3.1.5}$$

$$\tilde{f}_1(t) = \tilde{f}_1(t,\mu) = f_1(t) - N(t,\mu)\tilde{f}_2(t,\mu),\ \tilde{f}_2(t) = \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu) f_2(t),$$

$\tilde{x}^0 = \tilde{x}^0(\mu) = x^0 - \mu N(t_0,\mu)\tilde{z}^0$, $\tilde{z}^0 = \tilde{z}^0(\mu) = z^0 - H(t_0,\mu)x^0$, x^0, z^0 - given vectors.

For small values of the parameter μ, matrices $H(t) = H(t,\mu)$, $N(t) = N(t,\mu)$ is having a dimension $m \times n$, $n \times m$ are the solutions of singularly perturbed equations (see 1.1)

$$\mu\dot{H} + \mu H\tilde{A}_1(t) = A_3(t) + A_4(t)H,\ \ H(t_0) = H^0, \tag{3.1.6}$$

$$\mu\dot{N} - \mu\tilde{A}_1(t)N = -A_2(t) - N\tilde{A}_4(t),\ \ N(t_1) = N^1, \tag{3.1.7}$$

and $H(t,\mu) \to -A_4^{-1}(t)A_3(t)$, $N(t,\mu) \to -A_2(t)A_4^{-1}(t)$, at $\mu \to 0$.

Let $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ are normalized at the point $t = s$ $(t,s \in [t_0,t_1])$ transition matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1(t)\tilde{x}$, $\mu\dot{\tilde{z}} = \tilde{A}_4(t)\tilde{z}$.

We write the law of motion of the system (3.1.4) by the Cauchy formula:

$$\tilde{x}(t.\mu) = \Phi(t,t_0,\mu)\tilde{x}^0 + \int_{t_0}^{t}\Phi(t,s,\mu)[\tilde{B}_1(s,\mu)u(s,\mu) + \tilde{f}_1(s,\mu)]ds, \tag{3.1.8}$$

$$\tilde{z}(t.\mu) = \Psi(t,t_0,\mu)\tilde{z}^0 + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)[\tilde{B}_2(s,\mu)u(s,\mu) + \tilde{f}_2(s,\mu)]ds. \quad (3.1.9)$$

In view of (3.1.5) for the functions (3.1.8), (3.1.9) can be easily checked by limiting relations $\lim\limits_{\mu\to 0}\tilde{x}(t.\mu) = \bar{x}(t)$, $\lim\limits_{\mu\to 0}\tilde{z}(t.\mu) = -A_4^{-1}(t)\left(B_2(t)\bar{u}(t) + f_2(t)\right)$,

where $\bar{x}(t)$ - the vector of state "slow" subsystem (3.1.2). We introduce the following notation:

$$h_1(t,s,\mu) = \Phi(t,s,\mu)\tilde{B}_1(s,\mu), \;\; h_2(t,s,\mu) = \Psi(t,s,\mu)\tilde{B}_2(s,\mu). \quad (3.1.10)$$

Definition. Matrices $h_1(t,s,\mu)$ and $h_2(t,s,\mu)$, each line item form $r-$ dimensional vectors $h_{1i}(t,s,\mu)$, $h_{2j}(t,s,\mu)$ will be called the "slow and" fast "pulse transition matrices of the system (3.1.4) on the impact of $u_\mu(t) = u(t,\mu)$.

Comment. In the future, we will assume within the meaning of the impulse response matrix [86] that $h^{(1)}(t,s,\mu)$=0, $h_2(t,s,\mu)$=0 at $t < s$.

In view of (3.1.10) relations (3.1.8) and (3.1.9) can be written in the form

$$\tilde{x}(t,\mu) = \Phi(t,t_0\mu)\,x^0 + \int_{t_0}^{t}\Phi(t,s,\mu)\;\tilde{f}_1(s,\mu)ds + \int_{t_0}^{t}h^{(1)}(t,s,\mu)u(s,\mu)ds \quad (3.1.11)$$

$$\tilde{z}(t,\mu) = \Psi(t,t_0\mu)\,\tilde{z}^0 + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\;\tilde{f}_2(s,\mu)ds + \frac{1}{\mu}\int_{t_0}^{t}h^{(2)}(t,s,\mu)u(s,\mu)ds. \quad (3.1.12)$$

Substituting the boundary conditions $\tilde{x}(t_1,\mu) = \tilde{x}^1, \tilde{z}(t_1,\mu) = \tilde{z}^1$ in (3.1.10), (3.1.12), integral equations

$$\int_{t_0}^{t_1} h_{1i}(t_1,s,\mu)u(s,\mu)ds = \alpha_{1i}(\mu), \quad i=\overline{1,n} \tag{3.1.13}$$

$$\int_{t_0}^{t_1} h_{2j}(t_1,s,\mu)u(s,\mu)ds = \mu\alpha_{2j}(\mu), \; j=\overline{1,m} \tag{3.1.14}$$

where $\alpha_{1i}(\mu) = \tilde{x}_i^1 - \phi^{[i]'}(t_1,t_0,\mu)\,\tilde{x}_i^0 - \int_{t_0}^{t_1}\phi^{[i]'}(t_1,s,\mu)\,\tilde{f}_1(s,\mu)ds,$

$$\alpha_{2j}(\mu) = \tilde{z}_j^1 - \psi^{[j]'}(t_1,t_0,\mu)\,\tilde{z}_j^1 - \frac{1}{\mu}\int_{t_0}^{t_1}\psi^{[j]'}(t_1,s,\mu)\,\tilde{f}_2(s,\mu)ds,$$

$$x_i^1 = x_i^1(\mu) = x_i^1 - \mu N^{[i]'}(t_1,\mu)\tilde{z}_j^1, \;\; z_i^1 = z_i^1(\mu) = z_i^1 - H^{[i]'}(t_1,\mu)x_i^1,$$

x_i^1, z_j^1 - given numbers, $\phi^{[i]'}(t_1,t_0,\mu)$, $\psi^{[j]'}(t_1,t_0,\mu)$, $N^{[i]'}(t_1,\mu), H^{[j]'}(t_1,\mu)$ - vectors - lines whose components are formed from elements of the rows of the respective matrix $\Phi(t_1,t_0,\mu)$, $\Psi(t_1,t_0,\mu)$, $N(t_1,\mu)$, $H(t_1,\mu)$.

In accordance with the wording of the problem R_μ vector function $h_{1i}(t_1,t,\mu)$, $h_{2j}(t_1,t,\mu)$ $(i=\overline{1,n}; j=\overline{1,m})$ can be considered as elements of a space $M\{h_\mu(t)\}$, and the vector function $u_\mu(t)$ depicting the control as elements of the space $M^*\{u_\mu(t)\}$ adjoint to the $M\{h_\mu(t)\}$.

Then the problem R_μ is reduced to the problem of moments [42]. Left side of (3.1.13) and (3.1.14) are the linear operation $g\left[h_\mu(t)\right]$ performed on the elements $h_{\mu i}^{(1)}(t) = h_{1i}(t_1,t,\mu)$ $(i=\overline{1,n})$, $h_{\mu j}^{(2)}(t) = h_{2j}(t_1,t,\mu)$ $(j=\overline{1,m})$.

We formulate the problem of moments for the task R_μ :

Is required to find the linear operation $g\left[h_\mu(t)\right]$ certain space $M\left\{h_\mu(t)\right\}$, satisfying at predetermined elements $h_{\mu i}^{(1)}(t)$ $\left(i=\overline{1,n}\right)$, $h_{\mu j}^{(2)}(t)$ $\left(j=\overline{1,m}\right)$ conditions

$$g\left[h_{\mu i}^{(1)}(t)\right]=\alpha_{1i},\ \ i=\overline{1,n};\qquad g\left[h_{\mu j}^{(2)}(t)\right]=\mu\alpha_{2j},\ \ j=\overline{1,m} \tag{3.1.15}$$

and at the same norm $\rho_\mu^*[g]$, operations $g\left[h_\mu(t)\right]$, was the lowest of the possible.

Each linear operation that makes sense for functions $h_\mu(t)$, from $M\left\{h_\mu(t)\right\}$ is generated by a control $u_\mu(t)$, in integral form [42]. Therefore, interpreting the expression on the left side of (3.1.13) and (3.1.14) as a linear function of the operation generated $u_\mu(t)=u(t,\mu)$ can be replaced by the problem of determining $u_\mu(t)$ problem of moments, i.e. the task of determining the operation $g\left[h_\mu(t)\right]$ satisfying (3.1.15).

Then, in this case, according to the problem of moment [86], we need to find from family vector of function of the form

$$h(t_1,t,\mu)=l_1'h_1(t_1,t,\mu)+l_2'h_2(t_1,t,\mu) \tag{3.1.16}$$

function $h^0(t_1,t,\mu)$, at which the minimum

$$\begin{aligned}\rho_\mu^0&=\min_{l_1,l_2}\rho[l_1'h_\mu^{(1)}(t)+l_2'h_\mu^{(2)}(t)]\\&=\rho_\mu\left[l_1^{0'}h_\mu^{(1)}(t)+l_2^{0'}h_\mu^{(2)}(t)\right]=\rho_\mu^0[h_\mu^0(t)]\end{aligned} \tag{3.1.17}$$

at $l_1'\alpha_1 + \mu l_2'\alpha_2 = 1,$

where

$l_1' = (l_{11}, l_{12}, \dots, l_{1n}), \quad l_2' = (l_{21}, l_{22}, \dots, l_{2m}), \quad h_\mu^{(1)}(t) = h_1(t_1, t, \mu), \quad h_\mu^{(2)}(t) = h_2(t_{1,} t, \mu),$

$\rho_\mu[h_\mu(t)]$ - norm of a function in the space $\mathrm{M}\{h_\mu(t)\}$.

Function $h^0(t_1, t, \mu)$ we call the minimum function.

Minimum function $\bar{h}_0^0(t_1 t)$ for the task R_0 is determined from the condition

$$\rho_0^0 = \min_{l_1} [l_1' \bar{h}_0(t)] = \rho_0[l_1'^{0'} \bar{h}_0(t)] = \rho_0^0[\bar{h}^0(t)] \qquad (3.1.18)$$

at $l_1'\bar{\alpha}_1 = 1,$

where $\bar{h}_0(t) = \bar{h}_0(t_1, t) = \bar{\Phi}_0(t_1, t)\mathrm{B}_0(t), \quad \bar{\alpha}_1 = x^1 - \bar{\Phi}_0(t_1, t)x^0 - \int_{t_0}^{t_1} \bar{\Phi}_0(t_1, \tau) f_0(\tau) d\tau,$

$$h^0(t) = l^{0'} \bar{h}_0(t).$$

3.2 Control with Minimal Power

Consider the following problem of optimal control: it is required to find a control

$$u = u_\beta^0(t, \mu) \quad (u_\beta^0 \in R^1)$$

transforming system

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + B_1(t)u + f_1(t) \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + B_2(t)u + f_2(t) \end{aligned} \qquad (3.2.1)$$

from the initial state

$$x(t_0) = x^0, \quad z(t_0) = z^0 \tag{3.2.2}$$

to the final state

$$x(t_1) = x^1, \quad z(t_1) = z^1 \tag{3.2.3}$$

provided that the norm of

$$\|u\|_{L_\infty} = \max_{t_0 \le t \le t_1} |u(t,\mu)| \tag{3.2.4}$$

has reached the minimum value. Here $x \in R^n$, $z \in R^m, u \in R^1$, μ - small parameter.

Suppose that for the problem (3.2.1) - (3.2.4) the following conditions:

1^0. Matrices $A_i(t)$ $(i = \overline{1,4})$, $B_j(j = \overline{1,2})$ - uniformly bounded and uniformly continuous together with its derivatives at $t \in [t_0, t_1]$; matrix $A_4(t)$ nondegenerate, i.e. exist $A_4^{-1}(t)$.

2^0. Vectors $L_1(t), L_2(t), \ldots, L_n(t)$ are linearly independent, at least at one $t^* \in (t_0, t_1)$, i.e. $\sum_{i=1}^{n} v_i L_i(t^*) \neq 0$ at $\sum_{i=1}^{n} v_i^2 \neq 0$, where

$L_1(t) = B_0(t) = B_1(t) - A_2(t) A_4^{-1}(t) B_2(t),$

$$L_k(t) = A_0(t) L_{k-1} - \frac{dL_{k-1}}{dt}, \quad A_0(t) = A_1(t) - A_2(t) A_4^{-1}(t) A_3(t), \quad k = 2,3,\ldots,n;$$

3^0. At point $t = t_1$

$$rank\{B_2(t_1), A_4(t_1) B_2(t_1), \ldots, A_4^{m-1}(t_1) B_2(t_1)\} = m; \tag{3.2.5}$$

4^0. Roots $\lambda_i(t)$ of characteristic equation of the matrix $A_4(t)$ subject to inequality

$$\operatorname{Re}\lambda_i(t) \le -\gamma < 0 \quad (i=\overline{1,m}; \quad t \in [t_0,t_1]); \tag{3.2.6}$$

In case, when $A_i(t) \quad (i=\overline{1,4}), \quad B_j\,(j=\overline{1,2})$ - constant matrices, instead of the condition 1^0, 2^0 we make the following demands:

$$5^0.\ rank\{B_0, A_0B_0, \ldots, A_0^{n-1}B_0\} = n; \tag{3.2.7}$$

$$6^0.\ rank\{B_2, A_4B_2, \ldots, A_4^{m-1}B_2\} = m. \tag{3.2.8}$$

When the condition 1^0, 4^0 in the system (3.2.1) can make a complete separation of movements. After simple transformations (See Chapter 1), we obtain:

$$\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu), \tag{3.2.9}$$

$$\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu),$$

$$\tilde{x}(t_0) = \tilde{x}^0, \quad \tilde{z}(t_0) = \tilde{z}^0, \tag{3.2.10}$$

$$\tilde{x}(t_1) = \tilde{x}^1, \quad \tilde{z}(t_1) = \tilde{z}^1, \tag{3.2.11}$$

where $\tilde{A}_1(t,\mu) = A_1(t) - A_2(t)H(t,\mu), \quad \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$

$$B_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,), \quad \tilde{B}_2(t,\mu) = B_2 - \mu H(t,\mu)B_1(t),$$

$$\tilde{f}_1(t,\mu) = f_1(t) - N(t,\mu)\tilde{f}_2(t,\mu), \quad \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu) f_1(t),$$

$$\tilde{x} = x + \mu H(t,\mu)\tilde{z}, \qquad \tilde{z} = z - H(t,\mu)x,$$

$$\tilde{x}^\nu = x^\nu + \mu H(t_\nu,\mu)\tilde{z}^\nu, \qquad \tilde{z}^\nu = z^\nu - H(t_\nu,\mu)x^\nu, \quad \nu = 0,1\ldots;$$

Matrices $N(t,\mu)$ and $H(t,\mu)$ are solutions of the equations

$$\mu\dot{N} - \mu\tilde{A}_1 N = -A_2 - N\tilde{A}_4, \tag{3.2.12}$$

$$\mu\dot{H} + \mu H\tilde{A}_1 = A_3 + \tilde{A}_4 H. \tag{3.2.13}$$

Matrices elements N, H regular dependent parameters. As shown in Chapter 1, the matrices N and H of (3.2.12), (3.2.13) are uniquely determined, whereby instead of the problem (3.2.1) - (3.2.4) to consider the problem (3.2.9) - (3.2.11), (3.2.4). This problem reduces to the problem of moments: to find

$$\rho_\beta^0 = \min_{p,q} \int_{t_0}^{t_1} \left| \tilde{B}_1'(\sigma,\mu)\Phi'(t_1,\sigma,\mu)p + \tilde{B}_2'(\sigma,\mu)\Psi'(t_1,\sigma,\mu)q \right| d\sigma, \qquad (3.2.14)$$

on condition

$$C_1'p + \mu C_2'q = 1, \qquad (3.2.15)$$

where $\Phi(t,s,\mu)$ and $\Psi'(t,s,\mu)$ $(t_0 \le t \le t_1)$ - normalized at the point $s \in [t_0,t_1]$ the fundamental matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1 x, \quad \mu\dot{\tilde{z}} = A_4\tilde{z}$ respectively;

$$C_1 = \tilde{x}^1 - \Phi(t_1,t_0,\mu)\tilde{x}^0, \quad C_2 = \tilde{z}^1 - \Psi(t_1,t_0,\mu)\tilde{z}^0. \qquad (3.2.15a)$$

For the matrix $\Psi(t,s,\mu)$ the condition [42]

$$\left\|\Psi(t,s,\mu)\right\| \le C\exp\left({-\gamma_1(t-s)}/{\mu} \right)$$

for all $t,s \in [t_0,t_1], \quad t \ge s,$ when $C, \gamma_1 > 0 - const.$

If we assume that in some way solved the problem (3.2.14), (3.2.15) and thus found vectors $p = p^0, q = q^0$, then it will be known the minimum function

$$h_\beta^0(t,\mu) = \tilde{B}_1'(t,\mu)\Phi'(t_1,t,\mu)p^0 + \tilde{B}_2'(t,\mu)\Psi'(t_1,t,\mu)q^0 \qquad (3.2.16)$$

and the number of $p_\beta^0 > 0.$

According to the rule the problem of the moment [86], we have to define the desired optimal control $u = u_\beta^0(t,\mu)$ based on the maximum condition

$$\int_{t_0}^{t_1} h_\beta^0(t,\mu)u_\beta^0(t,\mu)dt = \max_u \int_{t_0}^{t_1} h_\beta^0(t,\mu)u(t,\mu)dt = 1 \qquad (3.2.17)$$

at $\max\limits_{t_0 \le t \le t_1} |u(t,\mu)| = \dfrac{1}{\rho_\beta^0}$ or else when $|u(t,\mu)| = \dfrac{1}{\rho_\beta^0}$.

Maximum integral in (3.2.17) will be achieved if each time t integrand $h^0(t,\mu)\,u(t,\mu)$ will be maximum. Thus, optimal control $u_\beta^0(t,\mu)$ must be determined from the condition [42]

$$h_\beta^0(t,\mu)u_\beta^0(t,\mu) = \max_u h_\beta^0(t,\mu)u(t,\mu) \qquad (3.2.18)$$

$$\text{at } |u(t,\mu)| \le \frac{1}{\rho_\beta^0} = \omega_\beta^0 \quad (t_0 \le t \le t_1). \qquad (3.2.19)$$

Since the system (1) non-singular, and hence the function $h_\beta^0(t,\mu)$ - a smooth [86], i.e. it is at a given time interval is zero only in a finite number of isolated values $t = t_j$. Then the solution of problem (3.2.18), (3.2.19) delivered by the expression

$$u_\beta^0(t,\mu) = \omega_\beta^0 sign(\tilde{B}_1'(t,\mu)\Phi'(t_1,t,\mu)p^0 + \tilde{B}_2'(t,\mu)\Psi'(t_1,t,\mu)q^0 . \qquad (3.2.20)$$

$$(t_0 \le t \le t_1)$$

Function $u_\beta^0(t,\mu)$ is defined everywhere except for a finite number of isolated values $t = t_j$, where the function standing under the sign of «sing» vanishes.

At $\mu = 0$ from (3.2.9) - (3.2.11) we obtain

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)\bar{u} + f_0(t), \quad \bar{x} = (t_\nu) = x^\nu, \quad \nu = 0;1\ldots; \qquad (3.2.21)$$

$$\bar{z} = -A_4^{-1}(t)\big(A_3(t)\bar{x} + B_2(t)\bar{u} + f_2(t)\big) \qquad (3.2.22)$$

where $A_0 = A_1 - A_2A_4^{-1}A_3, \quad B_0 = B_1 - A_2A_4^{-1}B_2, \quad f_0(t) = f_1 - A_2A_4^{-1}f_2.$

The resulting system is called a generating system [26]. It should be noted that the solution of (3.2.21), (3.2.22), (3.2.4) can not serve as a zero approximation of the problem (3.2.9) - (3.2.11), (3.2.4) as the in the vicinity of the boundary interval $[t_0, t_1]$ there may be finite number of isolated points (to m pieces), which must go through the process of switching control action. Also in this case, the issue of system switching from one state to another for rapid subsystem (3.2.9) remains open. So first of all we need to specify a system for which the optimal solution of the problem is well-defined zero approximation of the problem (3.2.9) - (3.2.11), (3.2.4).

Due to requirements regarding $A_i(t) \quad (i=\overline{1,4})$ system (3.2.1) (see the condition 1^0) solutions of equations (3.2.12) - (3.2.13) are limited and at $\mu \to 0$ will be performed:

$$\tilde{A}_1(t,\mu) \to A_0(t), \quad \tilde{A}_4(t,\mu) \to A_4(t), \quad \tilde{B}_1(t,\mu) \to B_0(t), \quad \tilde{B}_2(t,\mu) \to B_2(t),$$
$$\tilde{f}_1(t,\mu) \to f_0(t), \quad \tilde{f}_2(t,\mu) \to f_2(t).$$

Consider the system

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)\bar{u} + f_0(t), \quad \bar{x} = (t_v) = x^v \qquad (3.2.23)$$

$$\mu\dot{\bar{x}}_* = A_4(t_1)\bar{z}_* + B_2(t_1)\bar{u} + f_2(t), \quad \bar{z}_* = (t_v) = z_*^{\,v} \qquad (3.2.24)$$

where $z_* = \bar{z} + A_4^{-1}(t_1)A_3(t_1)\bar{x}, \quad z^v{}_* = z^v + A_4^{-1}(t_1)A_3(t_1)x^v, \quad v = 0;1\ldots$

The system (3.2.23), (3.2.24) approximates the system (3.2.9) - (3.2.11) with accuracy of the order of smallness $O(\mu)$ and it is obtained from (3.2.9) - (3.2.11) in the following approximations:

$$H(t,\mu)\approx H_0(t)=-A_4^{-1}(t)A_3(t),\quad N(t,\mu)\approx N_0(t)=-A_2(t)A_4^{-1}(t),\quad \tilde{A}_4(t_1+\tau\mu)\approx A_4(t_1),$$
$$\tilde{B}_2(t_1+\tau\mu)\approx B_2(t_1),\quad -\infty<\tau\le 0.$$

For the new system the minimum function takes the form

$$h_0^0(t,\mu)=h_0^0(t,\overline{p}^0,\overline{q}^0,\mu)=B_0'(t)\overline{\Phi}'(t_1,t)\overline{p}^0+B_2'(t_1)e^{-A_4'(t_1)\frac{t-t_1}{\mu}}\overline{q}^0, \tag{3.2.25}$$

where $\overline{\Phi}'(t_1,s)$ - the fundamental matrix of the homogeneous system $\dot{\overline{x}}=A_0\overline{x};\quad \overline{p}^0,\overline{q}^0$ - solutions extremal problem: find

$$\rho_0^0=\min_{\overline{p},\overline{q}}\int_{t_0}^{t_1}\left|h_0(t,\overline{p},\overline{q},\mu)\right|dt \tag{3.2.26}$$

on condition

$$\overline{C}_1'\overline{p}+\overline{C}_2'\overline{q}=1 \tag{3.2.27}$$

$$\overline{C}_1=x'-\overline{\Phi}(t_1,t_0)x^0,\quad \overline{C}_2=z_*^1-e^{-A_4(t_1)\tau_0}z_*^0=z_*^1+O(e^{-\gamma\tau_0})\approx z_*^1,\quad \tau_0=\frac{t_0-t_1}{\mu}. \tag{3.2.27a}$$

Optimal control (3.2.20) for this case is written as

$$\begin{gathered}u_0^0(t,\mu)=\omega_0^0 sign\left(\tilde{B}_0'(t,\mu)\Phi'(t_1,t)\overline{p}^0+\tilde{B}_2'(t_1)e^{-A_4(t_1)\tau}\overline{q}^0\right),\\ \tau=\frac{t-t_1}{\mu},t_0\le t\le t_1,\omega_0^0=\frac{1}{\rho_0^0}.\end{gathered} \tag{3.2.28}$$

Control (3.2.28) transfers the system (3.2.23), (3.2.24) from the initial states (x^0,z^0) to the final state (x^1,z^1) and it is the function of the relay. Minimum function $h_0^0(t,\mu)$ (3.2.25) may be zero in the vicinity of t_0,t_1, because it contains a function of boundary layer type, whereby the control (3.2.28) has a complete set of switching points, which is not always possible for the generating system (3.2.21), (3.2.22). Rewrite the equality (3.2.27) in the form of

$$\sum_{i=1}^{n}\bar{C}_i^{(1)}\bar{p}_1+\mu\sum_{k=1}^{m}\bar{C}_k^{(2)}\bar{q}k=1. \tag{3.2.29}$$

We now show one of the approximate methods of determining the optimal parameters $\bar{p}_i^0, \bar{q}_k^0 \quad (i=\overline{1,n}; \quad k=\overline{1,m})$.

Assuming that the vector $\bar{C}_1$ (3.2.27) satisfies $\bar{C}_n^{(1)}\neq 0$, from (3.2.29), we obtain

$$\bar{p}_n=\frac{1}{\bar{C}_n^{(1)}}(1-\sum_{i=1}^{n-1}\bar{C}_i^{(1)}\bar{p}_i-\mu\sum_{k=1}^{m}\bar{C}_2^{(2)}\bar{q}_k). \tag{3.2.30}$$

The following functions $h_0(t,\mu)=h_0(t,\bar{p},\bar{q},\mu)$ standing under the sign of the module in (3.2.26) can be represented as

$$B_0'(t)\bar{\Phi}'(t_1,t)p=K(t,t_1)\bar{p}=\sum_{i=1}^{n}K_i(t,t_1)\bar{p}_i, \tag{3.2.31}$$

$$B_0'(t)\bar{\Phi}'(t_1,t)p=K(t,t_1)\bar{p}=\sum_{i=1}^{n}K_i(t,t_1)\bar{p}_i, \tag{3.2.32}$$

where $\eta(\frac{t-t_1}{\mu})$ - function of the type of the boundary layer, in other words, for it has the estimate $\|\eta\|\leq C\exp\left(\gamma\left(\frac{t-t_1}{\mu}\right)\right), \quad C,\gamma>0-const$.

In view of (3.2.30) - (3.2.32) function $h_0(t,\mu)$ is written in the form

$$\begin{aligned} h_0(t,\mu)&=h_0(t,\tilde{p},\bar{q},\mu)\\ &=\frac{K_n(t_1,t)}{C_n^{(1)}}+\left(\tilde{K}'(t,t_1)-\frac{K_n(t,t_1)}{C_n^{(1)}}\tilde{C}_1'\right)\tilde{p}+\left(\eta'(\frac{t-t_1}{\mu})-\mu\frac{K_n(t_1,t)}{C_n^{(1)}}\tilde{C}_2'\right)\bar{q}, \end{aligned} \tag{3.2.33}$$

$$\tilde{K}'=(K_1,K_2,\ldots,K_{n-1}), \quad \tilde{p}'=(\bar{p}_1,\bar{p}_2,\ldots,\bar{p}_{n-1}),$$

where $\eta' = (\eta_1, \eta_2, \ldots, \eta_m)$, $\quad \bar{C}_1' = (\bar{C}_1^{(1)}, \bar{C}_2^{(1)}, \ldots, \bar{C}_{n-1}^{(1)})$,

$$\tilde{C}_2' = (\bar{C}_1^{(1)}, \bar{C}_2^{(2)}, \ldots, \bar{C}_m^{(2)}), \quad \bar{q}' = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_m).$$

In this case, the problem (3.2.26), (3.2.27) the relative minimum is reduced to the problem of the absolute minimum of the function

$$\rho_0(\tilde{p}, \bar{q}, \mu) = \int_{t_0}^{t_1} \left| h_0(t, \tilde{p}, \bar{q}, \mu) \right| dt. \tag{3.2.34}$$

Note that for the functions $h_0(t, \tilde{p}, \bar{q}, \mu)$, $\rho_0(\tilde{p}, \bar{q}, \mu)$ at $\mu \to 0$ holds the following limit relations:

$$\lim_{\mu \to 0} h_0(t, \tilde{p}, \bar{q}, \mu) = \bar{h}_0(t, \tilde{p}), \quad \lim_{\mu \to 0} \rho_0(t, \tilde{p}, \bar{q}, \mu) = \bar{\rho}_0(\tilde{p}), \tag{3.2.35}$$

where $\bar{h}_0(t, \tilde{p}) = \dfrac{K_n(t_1, t)}{C_n^{(1)}} + \left(\tilde{K}'(t, t_1) - \dfrac{K_n(t, t_1)}{C_n^{(1)}} \tilde{C}_1' \right) \tilde{p}, \quad \rho_0(\tilde{p}) = \int_{t_0}^{t_1} \left| \bar{h}_0(t, \tilde{p}) \right| dt.$

Numbers $\bar{p}_i = \bar{p}_i^0 \quad (i = \overline{1, n-1})$, $\quad \bar{q}_k = \bar{q}_k^0 \quad (k = \overline{1, m})$ determining the minimum function $h_0^0(t, \bar{p}^0, \bar{q}^0, \mu)$ will satisfy the system of equations

$$\begin{aligned} \frac{\partial \rho_0}{\partial \bar{p}_i} &= \int_{t_0}^{t_1} \left(K_i(t, t_1) \frac{K_n(t, t_1)}{C_n^{(1)}} \tilde{C}_1' \right) sign h_0(t, \tilde{p}, \bar{q}, \mu) dt = 0, \quad i = \overline{1, n-1}, \\ \frac{\partial \rho_0}{\partial \bar{q}_k} &= \int_{t_0}^{t_1} \left(\eta_i \left(\frac{t - t_1}{\mu} \right) \frac{\mu K_n(t, t_1)}{C_n^{(1)}} C_k^{(2)} \right) sign h_0(t, \tilde{p}, \bar{q}, \mu) dt = 0, \quad k = \overline{1, m}. \end{aligned} \tag{3.2.36}$$

As stated in [86] for the solution of this problem on the conditional minimum is considered the differential equations for the unknown parameters $l_k (k = \overline{1, M-1})$ (in this case, relatively $\bar{p}_i, \bar{q}_k$)

$$\frac{dl_i}{dv} = -\varepsilon \frac{\partial \rho(l_1, l_2, \ldots l_{M-1})}{\partial l_i}, \quad i = \overline{1, M-1} \tag{3.2.37}$$

where $\varepsilon > 0$ - coefficient of proportionality determining the "shutter" speed. In drawing up the differential equation (3.2.37) introduced a new parameter v, which is interpreted as the time counted at the point of the movement $l = \{l_i\}$ along "the descent of the curve" by some arbitrary point $\bar{l} = \{\bar{l}_i\}$ on the hyperplane $\sum_{l=1}^{M} C_i l_i = 1$ to the desired point $l^0 = \{l_i^0\}$, numerical integration by using the recurrence relation

$$l_i^{(j+1} = l_i^{(j)} - \varepsilon [\frac{\partial \rho(l_1, l_2, \ldots l_{M-1})}{\partial l_i}]_{l=l^{(j)}} \Delta v. \qquad (3.2.38)$$

From the second equation (3.2.36), we note that arbitrary $\dfrac{\partial \rho}{\partial \bar{q}_k} \quad (k = \overline{1, m})$ defined functions faster components $h_0(t, \tilde{p}, \bar{q}, \mu)$.

Then, in this case, will consider the following singularly perturbed differential equations respectively to $\bar{p}_i, \quad \bar{q}_k \quad \left(i = \overline{1, n-1}; \quad k = \overline{1, m}\right)$

$$\frac{d\bar{p}_i}{dv} = -\varepsilon \frac{\partial \rho_0}{\partial \bar{p}_i}, \quad \mu \frac{d\bar{q}_k}{dv} = -\varepsilon \frac{\partial \rho_0}{\partial \bar{q}_k}, \qquad (3.2.39)$$

where the partial derivatives $\dfrac{\partial \rho_0}{\partial \bar{p}_i}, \quad \dfrac{\partial \rho_0}{\partial \bar{q}_k}$ are defined by (3.2.36). For the numerical integration of the equation (3.2.39) can propose the following process of successive approximations. At $\mu = 0$ from (3.2.39) we obtain the reduced system

$$\frac{d\bar{p}_i}{dv} = -\varepsilon \frac{\partial \rho_0(\tilde{p})}{\partial \bar{p}_i}, \quad \left(i = \overline{1, n-1}\right). \qquad (3.2.40)$$

Equation (3.2.40) can be integrated numerically using the relation (3.2.38), identifying all $p_i = p_i^0 \quad \left(i = \overline{1, n-1}\right)$ and leaving them in the equation

$$\mu \frac{d\overline{q}_k}{dv} = -\varepsilon \frac{\partial \rho_0\left(\tilde{p}^0, \overline{q}\right)}{\partial \overline{q}_k}, \quad k = \overline{1, m} \qquad (3.2.41)$$

and making the substitution $\tau = \dfrac{t - t_1}{\mu}$ the right side of (3.2.41), we obtain:

$$\frac{d\overline{q}_k}{dv} = -\varepsilon \int_{\tau_0}^{0} \left[\eta_k\left(\sigma, \mu\right) - \mu \frac{b_n^{(0)} C_k^{(2)}}{C_n^{(1)}} \right] sign\tilde{h}_0\left(\sigma, \mu\right) d\sigma, \qquad (3.2.42)$$

where

$$\begin{aligned} \tilde{h}(\sigma, \mu) &= h_0\left(\sigma\mu + t_1, \tilde{p}^0, \overline{q}, \mu\right) = h_0\left(t_1, \tilde{p}^0, \overline{q}\right) + h_0'(t_1, \tilde{p}, q)\sigma\mu + \ldots \approx \\ &\approx \sum_{i=1}^{n-1} \left[b_i^{(0)} - \frac{C_i^{(1)}}{C_n^{(1)}} b_n^{(0)} \right] \overline{p}_i^0 + \frac{1}{C_n^{(1)}} b_n^{(1)} + \sum_{k=1}^{m} \left[\eta_k(\sigma, \mu) - \mu \frac{b_n^{(0)} C_k^{(2)}}{C_k^{(1)}} \right] \overline{q}_k, \end{aligned} \qquad (3.2.43)$$

where $\overline{p}^0 = \left(\overline{p}_1^0, p_2^0, \ldots, \overline{p}_{n-1}{}^0\right), \quad b_i^0\left(i = \overline{1, n}\right)$ - vector components $B_0 = B_0\left(t_1\right)$..

Now, again using the relations (3.2.44) can be integrated into the equation (3.2.42).

After the necessary calculations will be known $\overline{q}_k{}^0\left(k = \overline{1, m}\right)$. Number $\rho_0{}^0$ calculated using the formula (3.2.34). Number $\overline{\omega}_0{}^0 = \dfrac{1}{\rho_0{}^0}$ characterizes the amplitude of the control action.

Suppose now that for sufficiently small $\mu\left(0 < \mu < \mu_0\right)$ and the boundary points $\left(x^0, z^0\right)$, $\left(x^1, z^1\right)$ following conditions are met:

a) the angles of intersection of the graph of (3.2.25)

$$h_0^{\,0}\left(t,\mu\right)=h_0^{\,0}\left(t,\tilde{p}^{\,0},\overline{q}^{\,0},\mu\right)=B_0'\left(t\right)\bar{\varPhi}_0'\left(t,t_1\right)\overline{p}^{\,0}+B_2'\left(t_1\right)e^{-A_4^1\left(t_1\right)\left(t-t_1\right)/\mu}\overline{q}^{\,0} \quad (3.2.44)$$

to t axis nonzero;

b) Jacobian $\dfrac{\partial\left(\dfrac{\partial\rho_0}{\partial\tilde{p}},\dfrac{\partial\rho_0}{\partial\overline{q}}\right)}{\partial\left(\tilde{p},\overline{q}\right)}$ at $p=\overline{p}^{\,0},\overline{q}=\overline{q}^{\,0}$ nonzero.

When the above conditions function $h_0\left(t,\mu\right)=h_0\left(t,\tilde{p},\overline{q},\mu\right)$ vanishes only for a finite number of isolated points in time $t=t_j\left(\overline{p},\overline{q},\mu\right)\left(j=\overline{1,s}\right)$, are defined as a function of the magnitude unambiguous $\overline{p}_i$, $\overline{q}_k$, $\left(i=\overline{1,n};\quad k=\overline{1,m}\right)$ and the partial derivatives $\dfrac{\partial t_j}{\partial\overline{p}_i}$, $\dfrac{\partial t_j}{\partial\overline{q}_k}$ at $\overline{p}_i=\overline{p}_i^{\,0}$, $\overline{q}_k=q_k^{\,0}$ $\left(i=\overline{1,n};k=\overline{1,m}\right)$, exist as it follows from the implicit function theorem.

Even under this condition from the same implicit function theorem implies that $\overline{p}_i^{\,0}\left(i=\overline{1,n}\right)$ are continuously differentiable functions on $C_i^{(1)}\left(i=\overline{1,n}\right)$, and $\overline{q}_k^{\,0}\left(k=\overline{1,m}\right)$ are continuously differentiable functions on $C_i^{(1)}$ $\left(i=\overline{1,n}\right)$, $C_k^{(2)}$ $\left(k=\overline{1,m}\right)$. Then for small changes $\Delta C_i^{(1)}$, $\Delta C_k^{(2)}$ will be small changes in variables $\overline{p}_i^{\,0}$, $\overline{q}_k^{\,0}$, where are the estimates:

$$\left|\Delta\overline{p}_i^{\,0}\right|\le r_1\left\|\Delta c_1\right\| \quad (3.2.45)$$

$$\left|\Delta\overline{q}_k^{\,0}\right|\le r_2\left(\left\|\Delta c_1\right\|+\left\|\Delta c_2\right\|\right) \quad (3.2.46)$$

$$\left|\Delta\omega_0^{\,0}\right|\le r_3\left(\left\|\Delta c_1\right\|+\left\|\Delta c_2\right\|\right) \quad (3.2.47)$$

where r_i - positive numbers.

Changing the minimum function $h_0^0(t,\mu)$ depends not only on changes in the values $\overline{p}_i^o$, $\overline{q}_k^0$, and members of unregistered matrix decomposition $\Phi(t,t_1\mu)$ and $\psi(t,t_1\mu)$. Imagine matrices $\Phi(t,s,\mu)$, $\psi(t,s,\mu)$ in the shape of:

$$\Phi(t,s,\mu) = \overline{\Phi}(t,s) + \mu\phi(t,s,\mu) \tag{3.2.48}$$

$$\psi(t,s,\mu) = e^{A_4(t)(t-s)/\mu} + \xi(t,s,\mu) \tag{3.2.49}$$

It is easy to show that for sufficiently small values $\mu < \mu^0$ functions φ and ξ satisfy the inequalities

$$\|\phi(t,s,\mu)\| \le d_2 C^2 (e^{t-s} - 1) e^{-m(t-s)} \tag{3.2.50}$$

$$\|\xi(t,s,\mu)\| \le C(e^{d_1 C(t-s)} - 1) e^{-\frac{\gamma(t-s)}{\mu}} \tag{3.2.51}$$

where $m > 1, d_1, d_2, C -$const, $0 < \mu \le \mu^0$, $\quad \mu^0 = \min\left\{\frac{1}{d_2 C}, \frac{\gamma}{d_1 C}\right\}$.

Then, for any vectors p and q hyperplane of (3.2.27), in particular in $p = \overline{p}^0, q = \overline{q}^0$ function $h_\beta(t,p,q,\mu) \quad (t_0 \le t \le t_1)$ can be represented as

$$h_\beta(t,\overline{p}^0,\overline{q}^0,\mu) = h_0(t,\overline{p}^0,\overline{q}^0,\mu) + O(\mu + e^{\gamma\tau}), \tag{3.2.52}$$

where $\tau = \dfrac{t - t_1}{\mu} < 0, \gamma > 0 -$const.

This means that the function $h_\beta(t,\overline{p}^0,\overline{q}^0,\mu)$ has as many zeros as had $h_0(t,\overline{p}^0,\overline{q}^0,\mu)$. These zeros are placed with precision $O(\mu + e^{\gamma\tau})$ (the distance from the boundary point $t = t_1$ with precision $O(\mu)$), near the respective zeros

$h_0(t,\bar{p}^0,\bar{q}^0,\mu)$. In view of (3.2.48) and (3.2.49) from (3.2.15a), (3.2.27a), we obtain

$$\begin{aligned}\Delta C_1 = \mu(N(\underline{t_1,\mu})\tilde{z}^1 - \phi(t_1,t_0,\mu)x^0 \\ - \Phi_0(t_1,t_0)N(t_0,\mu)\tilde{z}^0 - \mu\phi(t_1,t_0,\mu)N(t_0,\mu)z^0,\end{aligned} \tag{3.2.53}$$

$$\begin{aligned}\Delta C_2 = \mu(e^{-A_4(t_1)\tau_0} \cdot H_1(t_0,\mu)x^0 - H_1(t_1,\mu)x') - \xi(t_1,t_0,\mu)\tilde{z}^0 \\ \approx -\mu H_1(t_1,\mu)x^1,\end{aligned} \tag{3.2.54}$$

where $H_1(t,\mu)$ - limited function, which appears from the relation:

$$H(t,\mu) = -A_4^{-1}(t)A_3(t) + \mu H_1(t,\mu),\ \tau_0 = \frac{t_0 - h}{\mu} \le 0.$$

Considering that the matrices N,H limited and functions φ,ξ satisfy (3.2.50), (3.2.51), from (3.2.45) - (3.2.47), we obtain:

$$\left|\Delta\bar{p}_i^0\ \right| \le m_1\mu,\ \left|\Delta\bar{q}_k^0\right| \le m_2\mu,\ \left|\Delta\omega_0^0\right| \le m_3\mu,\ \ m_1,m_2,m_3,\gamma > 0 - \text{const.} \tag{3.2.55}$$

These estimates suggest that for all t, except for a set Q values t, a measure which satisfy the inequality

$$\sigma(Q) \le m_4\mu\,. \tag{3.2.56}$$

Control $u_0^0(t,\mu)$ differs from the optimal control $u_\beta^0(t,\mu)$ the original problem with the accuracy $O(\mu)$, i.e. will be performed the following inequality:

$$\left|u_\beta^0 - u_0^0\right| = \left|\Delta u_0^0\right| \le m_5\mu,\ \ m_4,m_5 - \text{const.} \tag{3.2.57}$$

Hence we have the following conclusion:

Theorem 3.2.1. If the conditions a), b), then for sufficiently small values $\mu < \mu_0 \left(\mu_0 = \min\left\{ \frac{1}{d_2 C}, \frac{\gamma}{d_1 C} \right\} \right)$

1) optimal control $u_\beta(t,\mu)$ (3.2.20) can be approximated by a control $u_0^0(t,\mu)$ (3.2.28) with precision $O(\mu)$;

2) at $\mu \to 0$ both control - $u_{\beta,} u_0^0$ one tends to the same limit, i.e. $\lim\limits_{\mu\to 0} u_\beta(t,\mu) = \lim\limits_{\mu\to 0} u_0^0(t,\mu) = u^*(t)$.

This item is the optimal solution for the problem of the generating system (3.2.21), the order of which is lower than (3.2.1). All these statements are true for all t, except for a set Q values, a measure which is of the order of smallness $O(\mu)$.

In conclusion, it should be noted that the above method is easily offended by vector control case.