Chapter 4

Research Tasks of Optimal Control of Dynamic Processes Economy

4.1 Decomposition of an Extreme Problem of Interbranch Balance

The dynamic model of interbranch balance describes by a system of singularly perturbed differential equations

$$\begin{aligned} \dot{x} &= \left(E_k - A_1(t)\right)x - A_2(t)z - w^{(1)}, \\ \mu\dot{z} &= -A_3(t)x + \left(E_{n-k} - A_4(t)\right)z - w^{(2)}, \end{aligned} \tag{4.1.1}$$

where E_k, E_{n-k} – identity matrix sizes of $k \times k$, $(n-k)\times(n-k)$ respectively; $w^{(1)}, w^{(2)}$ – vectors with dimensions of the final product k and $n-k$ respectively.

The formulation of an extreme problem for the system (4.1.1) is similar [44], leads to some complex restrictions that in solving the problem must take into account that creates certain difficulties and, ultimately, the inability to obtain effectively implemented algorithm. Therefore, it is necessary to replace the system (4.1.1) is equivalent to the system, which are separated by slow and fast position.

Consider the system

$$\begin{aligned} \dot{x} &= \left(E_k - A_0(t)\right)x - u^{(1)}, \\ \mu\dot{\tilde{z}} &= \left(E_{n-k} - A_4(t)\right)\tilde{z} - u^{(2)}, \end{aligned} \tag{4.1.2}$$

where

$$\begin{gathered} A_0 = A_1 - A_2\left(E_{n-k} - A_4\right)^{-1}A_3, \quad \tilde{z} = z + \left(E_{n-k} - A_4\right)^{-1}A_3x, \qquad u^{(2)} = w^{(2)}, \\ u^{(1)} = w^{(1)} - A_2\left(E_{n-k} - A_4\right)^{-1}w^{(2)}. \end{gathered} \tag{4.1.3}$$

As shown in Section 1.2, such a system may change the original system, since it has all the properties of the original and is an oversimplification. Given the initial conditions and parameters known to control the decision of extreme problems for the system (4.1.2) can serve as an approximate solution of the extremal problem for the system (4.1.1) with precision $O(\mu)$.

We take as the control parameters the flow of final consumption w_j. The vector flow of consumption must obey certain natural limitations. The simplest kind of restrictions can be expressed by the following requirements:

1. In each branch the flow of consumption w_j can not be less than the specified minimum permissible value $\psi_j \geq 0,$ i.e. $w_j \geq \psi_j, \quad j = 1, 2, ... n.$ Then naturally function $u^{(1)}(t), u^{(2)}(t)$ obtain following restrictions:

$$\begin{aligned} &u^{(1)} \geq \psi^{(1)} - A_2\left(E_{n-k} - A_4\right)^{-1}\psi^{(2)}, \\ &u^{(2)} \geq \psi^{(2)}, \end{aligned} \tag{4.1.4}$$

where $u^{(1)}, \psi^{(1)} - k$ - dimensional, $u^{(2)}, \psi^{(2)} - (n-k)$ - dimensional vector functions.

The minimum allowable flow of consumption $\psi_j(t)$ can be determined by the rate consumption of production and of the population.

2. Vector flow accumulation looks: $\overline{s} = \begin{pmatrix} E_k & 0 \\ 0 & \mu E_{n-k} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \geq 0.$

In this case, the system (4.1.2), we have:

$$\psi^{(1)} - A_2\left(E_{n-k} - A_4\right)^{-1}\psi^{(2)} \leq u^{(1)} \leq \left(E_k - A_0\right)x, \tag{4.1.5}$$

$$\psi^{(2)} \leq u^{(2)} \leq \left(E_{n-k} - A_4\right)\tilde{z}. \tag{4.1.6}$$

Thus, we must assume that the control functions subsystems (4.1.2) associated restrictions (4.1.5) and (4.1.6).

The optimality criterion, we can select a functional:

$$J = \int_{t_0}^{t_1} \left[\left(q^{(1)}, w^{(1)} \right) + \left(q^{(2)}, w^{(2)} \right) \right] dt \, . \tag{4.1.7}$$

where $\left(q^{(1)}, w^{(1)} \right) = \sum_{i=1}^{k} q_i w_i$, $\left(q^{(2)}, w^{(2)} \right) = \sum_{j=k+1}^{n} q_j w_j$, $q_i \geq 0$ $i = 1, 2, ..., n$ - a nonnegative decreasing function. The functional (4.1.7) represents the total amount of increase of social welfare for the period $\left[t_0, t_1 \right]$ [50]. Of course, we must strive to build such functions $w_j \left(t \right)$, $j = 1, 2, ..., n,$ that deliver the greatest value of this functionality.

Now, taking into account the relations (4.1.3) functional (4.1.7) may be represented as:

$$J = \int_{t_0}^{t_1} \left(q^{(1)}, u^{(1)} \right) dt + \int_{t_0}^{t_1} \left(\tilde{q}^{(2)}, u^{(2)} \right) dt = J^{(1)} + J^{(2)}, \tag{4.1.8}$$

where $\left(q^{(1)}, u^{(1)} \right) = \sum_{i=1}^{k} q_i^{(1)} u_i^{(1)}$, $\left(\tilde{q}^{(2)}, u^{(2)} \right) = \sum_{i=k+1}^{n} \tilde{q}_i^{(2)} u_i^{(2)}$, $\tilde{q}_i^{(2)} = \sum_{j=1}^{k} q_j^{(1)} r_{ji} + q_i^{(2)}$,

r_{ji} – elements of the matrix $R = A_2 \cdot \left(E_{n-k} - A_4 \right)^{-1}$.

It should be noted that the slow and fast sub system (4.1.2) are not connected, and it is possible to consider each separately. Furthermore, as the functional J represents the sum. The same function $u^{(1)} = u_0^{(1)}$ delivers the extreme values of the functionals J and $J^{(1)}$ with the following restrictions:

a) the rate of production growth $\dot{x} = \left(E_k - A_0 \left(t \right) \right) x - u^{(1)}$,

b) on the initial and final output of production $x(t_0)=x^0, \qquad x(t_1)=x^1.$

This is due to the fact that the functional $J^{(2)}$ in Formula (4.1.8) does not depend on the choice of $u^{(1)}$.

Similarly, the same function $u^{(2)}=u_0^{(2)}$ delivers the extreme values of the functionals J and $J^{(2)}$ with restrictions:

a) the rate of production growth $\mu\dot{\tilde{z}}=\left(E_{n-k}-A_4\right)\tilde{z}-u^{(2)}$;

b) on the initial and final output $\tilde{z}(t_0)=\tilde{z}^0, \qquad \tilde{z}(t_1)=\tilde{z}^1.$

Thus, the original optimization problem is divided into two problems that have smaller dimension systems and fewer restrictions, which are considered in the optimization process that shows the effectiveness of the proposed algorithm with the position of its use in practice.

4.2 Solution of Singularly Perturbed Problem on Optimal Economic Growth

To get one of the possible solutions to this problem

$$\frac{1}{2}b\int_{t_0}^{t_1}e^{-2\delta(t-t_0)}\left(c(t)-\frac{a}{b}\right)^2dt=\int_{t_0}^{t_1}u^2(t)dt=\|u\|^2_{L_2[t_0,t_1]}, \tag{4.2.1}$$

$$\mu\dot{x}=-\lambda_1x+F(x)-u\,, \tag{4.2.2}$$

$$x(t_0)=\sqrt{\frac{b}{2}}k_0=x_0\,, \tag{4.2.3}$$

$$x(t_1)=\sqrt{\frac{b}{2}}e^{-\delta(t_1-t_0)}k_1=x_1\,, \tag{4.2.4}$$

where the function $f(k)$ has property $f(\alpha k) = \alpha f(k)$ in [85] assume that consumption of worker's and productive reserves is not changed in time. Let us take this assumption for this problem.

In addition, we believe that the rate of discount δ constant and positive, and its value is considered to be quite large, which indicates a greater preference for the useful life of loved ones [82, 85].

If put $\dot{x} = 0$, then from (4.2.2) we get:

$$F(x) - \lambda_1 x - u = 0 \tag{4.2.5}$$

or

$$f(k) - \lambda_1 k - c = 0\,. \tag{4.2.6}$$

If the value k during the time of transition retains its constant value, the control parameter c does not change respectively to k .

By then, the condition $\frac{dc}{dk} = 0$ follows that

$$f'(k) = \lambda_1. \tag{4.2.7}$$

Under the made assumptions respectively to the production function $f(k)$ from (4.2.6), (4.2.7) can only determine the value $k = k^*, \quad c = c^* = f(k^*) - \lambda_1 k^*$, which satisfy the following inequality:

$$0 < c^* < f(k^*)\,. \tag{4.2.8}$$

Balance at $k(t) = k^*$, $c(t) = c^*$ meets all the necessary conditions, except for the boundary conditions

$$k(t_0) = k_0, \tag{4.2.9}$$

$$k(t_1) = k_1. \tag{4.2.10}$$

These values are determined balanced growth mode [85, 93]. With $\mu \to 0$ values k^*, c^* have limits, i.e. $\lim_{\mu\to 0} k^* = \bar{k}^*$, $\lim_{\mu\to 0} c^* = \bar{c}^*$, where $\bar{k}^*, \bar{c}^*$ – solutions of the equations of the form $f'(k) = \varepsilon$, $f(k) - \varepsilon k - c = 0$. On the other hand, the condition $\dot{x}(t) = 0$ we have:

$$\dot{k} - \delta k = 0. \tag{4.2.11}$$

To associate a non-zero solution of (4.2.11) with the point of rest $k = k^*$ we fix one value $t_* \in [t_0, t_1]$. As described above, for any $t_* \in [t_0, t_1]$ is the equality:

$$k(t_*) = k^*. \tag{4.2.12}$$

The solution of the task (4.2.11), (4.2.12) has the form:

$$k(t, k^*) = e^{\delta(t - t^*)} k^*. \tag{4.2.13}$$

This decision at $t \to t^*$ tends to the point of rest k^*. In view of (4.2.13) from the relation $x(t) = \sqrt{\frac{b}{2}} e^{-\delta(t-t_0)} k(t)$,

$$F(x) = f\left(\sqrt{\frac{b}{2}} e^{-\delta(t-t_0)} k(t)\right) - \frac{a}{\sqrt{2b}} e^{-\delta(t-t_0)} = f(x) - \frac{a}{\sqrt{2b}} e^{-\delta(t-t_0)} \tag{4.2.14}$$

we define $x = x^*$, $u = u^*$:

$$x^* = \sqrt{\frac{b}{2}} \cdot e^{-\delta(t^* - t_0)} \cdot k^*, \quad u^* = \sqrt{\frac{b}{2}} e^{-\delta(t^* - t_0)} \left(c^* - \frac{a}{b}\right). \tag{4.2.15}$$

These values satisfy the equation (4.2.5). This point $\tilde{x}^* = \dfrac{F(x^*) - u^*}{\varepsilon}$ is a rest point "connected systems" [20] (in this case one equation)

$$\frac{d\tilde{x}}{d\tau} = -\varepsilon_1 \tilde{x} + F(x^*) - u^*, \quad \tau \ge 0, \tag{4.2.16}$$

$\tilde{x}(0) = x_0$, (or when $\tau \le 0$)

$$\tilde{x}(0) = x_1. \tag{4.2.17}$$

The point of rest $\tilde{x}^*$ is asymptotically stable in the Lyapunov $\tau \to \infty$. Now the equation (4.2.2) can be rewritten as:

$$x(t,\mu) = e^{-\lambda_1 \frac{t-t_0}{\mu}} x_0 + \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1 \frac{t-s}{\mu}} F(x(s))ds - \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1 \frac{t-s}{\mu}} u(s)ds, \tag{4.2.18}$$

where $\lambda_1 = \lambda_1(\mu) = \varepsilon + \mu(n+\delta)$.

If put in (4.2.2) $F(x(t)) \approx F(x^*)$, then in the known $u = u(t)$ phase coordinates $x(t) = x(t,\mu)$ taking into account the boundary conditions is approximately determined by the formula:

$$x(t,\mu) \approx e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)} x_0 + \frac{F(x^*)}{\lambda_1}\left(1 - e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)}\right) - \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1\left(\frac{t-s}{\mu}\right)} u(s)ds. \tag{4.2.19}$$

We agree that the boundary points $x(t_0) = \sqrt{\frac{b}{2}}k_0 = x_0, \quad x(t_1) = \sqrt{\frac{b}{2}}e^{-\delta(t_1-t_0)}k_1 = x_1$ belong to the domain of influence point of the rest $\tilde{x}^*$. This means that the solution of equation (4.2.16) with the initial condition $\tilde{x}(0) = x_0$ exists for $\tau \ge 0$ and tends to the point of rest $\tilde{x}^*$ at $\tau \to +\infty$, and another solution (4.2.16) with the initial $\tilde{x}(0)=x_1$ also exists for $\tau \le 0$ and tends to the point of rest $\tilde{x}^*$ at $\tau \to -\infty$.

Note. With $\tau \le 0$ from (4.2.17) we obtain the equation with reverse time $\frac{d\tilde{x}}{d\tau} = \varepsilon \tilde{x} - F\left(x^*\right) + u^*, \qquad \tau \le 0.$

In this case, we are interested in is the solution of the problem (4.2.1) (4.2.2) - (4.2.4) which, when $\mu \to 0$ passes to the solution of the problem generator. By (4.2.5), (4.2.15), (4.2.19) of the difference $x(t,\mu)-x^*$ define the following formula:

$$x(t,\mu) \approx e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)}(x_0 - x^*) + x^* - \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1\left(\frac{t-s}{\mu}\right)} u(s)\,ds. \tag{4.2.20}$$

We define control as follows:

$$u^0(t) = u^0(t,\mu) = V\left(\frac{t-t_0}{\mu}\right), \qquad 0 \le \frac{t-t_0}{\mu} \le \frac{t_1-t_0}{\mu} < +\infty . \tag{4.2.21}$$

At $t=t_1$ из (4.2.7) we obtain the following relationship of moment:

$$\alpha_1 = \int_0^{\tau_1} e^{-\lambda_1(\tau_1-\tau)} V(\lambda)\,d\lambda, \tag{4.2.22}$$

where $\alpha_1 = -x_1 + e^{-\lambda_1\tau_1}x_0 + x^*\left(1-e^{-\lambda_1\tau_1}\right)$, $\tau_1 = \frac{t_1-t_0}{\mu}$.

Then the function $u^0(t)=V(\tau)$ satisfying the relationship of moment (4.2.22), with a minimum rate is:

$$V(\tau) = \frac{2\lambda_1\alpha_1 e^{-\lambda_1(\tau_1-\tau)}}{1-e^{-2\lambda_1\tau_1}} . \tag{4.2.23}$$

It should be noted that when $\tau_1 - \tau \to \infty$ ($\mu \to 0$) from (4.2.23) we get the following limit relation:

$$\lim_{\substack{\tau_1 - \tau \to \infty \\ (\mu \to 0)}} V(\tau) = 0. \tag{4.2.24}$$

At the same time, this control $u^0(t,\mu)$ (4.2.21) is optimal in the sense of the task and the corresponding phase coordinates can be written as:

$$x^0(t,\mu) = \frac{F(x^*) - u^*}{\lambda_1} + e^{-\lambda_1 \tau}\left(x_0 - \frac{F(x^*) - u^*}{\lambda_1}\right) - \frac{\alpha_1 e^{-\lambda_1(\tau_1 - \tau)}\left(1 - e^{-2\lambda_1 \tau}\right)}{1 - e^{-2\lambda_1 \tau_1}}, \quad 0 \le \tau \le \tau_1$$

or

$$x^0(t,\mu) = e^{-\lambda_1 \tau} x_0 + \tilde{x}^*\left(1 - e^{-\lambda_1 \tau}\right) - \frac{e^{-\lambda_1(\tau_1 - \tau)}\left(1 - e^{-2\lambda_1 \tau}\right)}{1 - e^{-2\lambda_1 \tau_1}}\left(-x_1 + e^{-\lambda_1 \tau_1} x_0 + x^*\left(1 - e^{-\lambda_1 \tau_1}\right)\right), \quad . \tag{4.2.25}$$

$0 \le \tau \le \tau_1$

Function $x(t,\mu)$ (4.2.25) satisfies all the boundary conditions (4.2.3) (4.2.4) at $\mu \to 0 \quad (\tau \to \infty)$

$$x^0(t,\mu) \to \frac{F(x^*) - u^*}{\varepsilon} = \tilde{x}^*. \tag{4.2.26}$$

This means that the point of rest x^* asymptotically stable in the Lyapunov $\tau \to \infty$. The limit relation (4.2.26) can be written in the form:

$$\lim_{\mu \to 0} \sqrt{\frac{b}{2}} e^{-\delta(t - t_0)} k_\mu^0(t,\mu) = \sqrt{\frac{b}{2}} e^{-\delta(t^* - t_0)} \cdot \frac{f(k^*) - c^*}{\varepsilon}, \tag{4.2.27}$$

where

$$k_\mu^0(t,\mu) = e^{-\delta(t^*-t)}\frac{f(k^*)-c^*}{\lambda_1}$$
$$+e^{\delta(t-t_0)}\left[e^{-\lambda_1\tau}\left(k_0 - e^{-\delta(t^*-t_0)}\frac{f(k^*)-c^*}{\lambda_1}\right) - \frac{\alpha_1 e^{-\lambda_1(\tau_1-\tau)}\left(1-e^{-2\lambda_1\tau}\right)}{\left(1-e^{-2\lambda_1\tau_1}\right)}\right]. \quad (4.2.28)$$

Function $k^0(t,\mu)$ satisfies all the boundary conditions (4.2.9), (4.2.10), and for its satisfies the following limit relation:

$$\lim_{\substack{\mu\to 0\\ t\to t^*}} k_\mu^0(t,\mu) = \frac{f(k^*)-c^*}{\varepsilon}.$$

In this case, "highway" generates no equilibrium line, and generates a function of the form:

$$\bar{k}_0^*(t,0) = e^{-\delta(t^*-t)}\cdot\frac{f(k^*)-c^*}{\varepsilon}, \qquad t^*\in[t_0,t_1]. \qquad (4.2.29)$$

The optimal trajectory, leaving the start point is sent to the "highway" and for quite some time are close to the line (for sufficiently small μ) and goes with it to achieve the desired end state (see. Fig. 4.2.2).

We construct the optimal trajectories $x(t,\mu)$ (4.2.25), $k_\mu^0(t,\mu)$ (4.2.28) и corresponding line (Fig. 4.2.1, 4.2.2) with the following data [85]: the time $t\in[0;21]$; the growth rate of the labor force $n=0{,}0053$; depreciation rate $\varepsilon=\frac{1}{13}\approx 0{,}0769$; discount coefficient $\delta=0{,}10$; a small positive parameter $\mu=0{,}5$ $\lambda=\varepsilon+n=0{,}0822$, $\lambda_1=\varepsilon+\mu(n+s)=0{,}1295$.

The calculations used the following data:

factor of production costs $a = 0,6$, $a_0 = 2{,}189$, incremental coefficient fond intensity $b = \frac{a_0}{1-a} = 5,4725$, the elasticity of manufacture production assets $\alpha = 0,249$, the elasticity of labor issue $\beta = 0,751$. The initial and final state of the capital-set values for $t = 0$ and $t = 21$: $k(0) = k_0 = 1{,}8682$, $k(21) = k_1 = 3{,}7772$, time to highway $t^* = 3$, corresponding capital armament $k^* = 2,0431$.

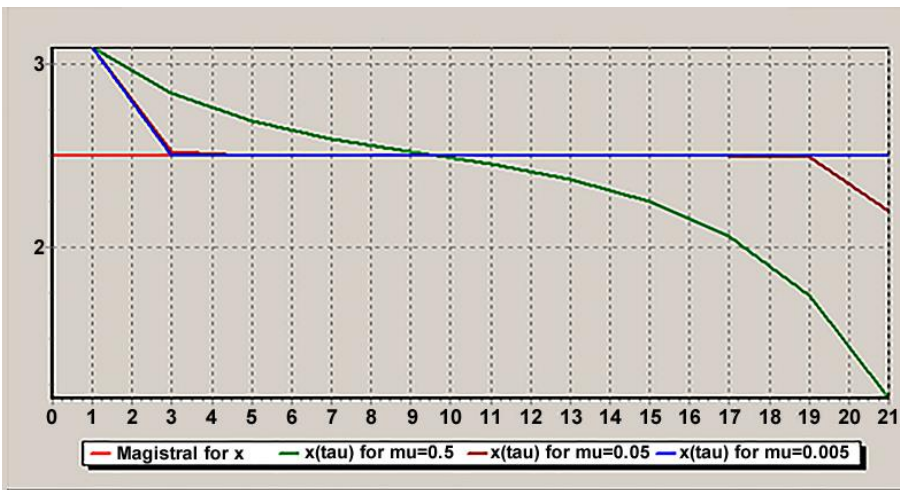

Fig. 4.2.1 *The optimum trajectory for capital intensity of worker* $x(t,\mu)$.

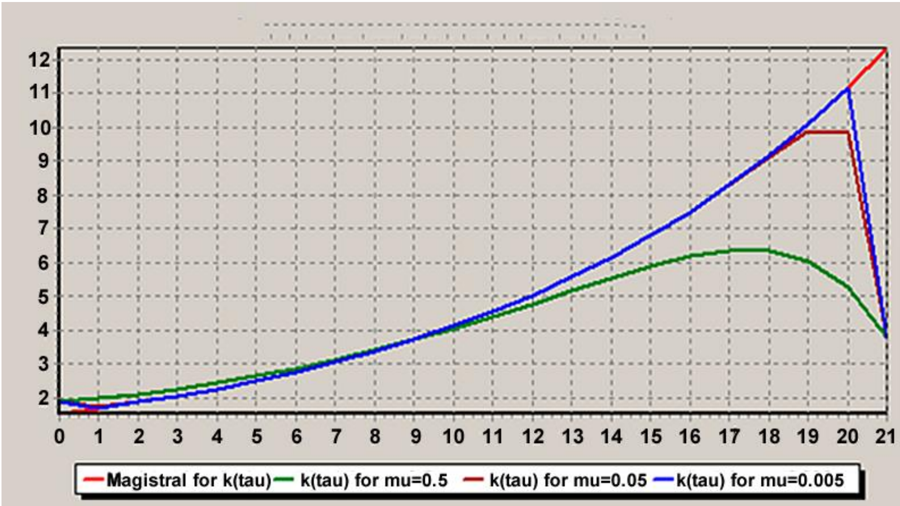

Fig. 4.2.2 *The optimum trajectory for capital intensity of worker* $k(t,\mu)$.

4.3 Control in Single-Commodity Macroeconomic Dynamic Model for Different Optimality Criteria

In here solve two problems of optimal distribution of gross product which illustrates the results obtained in [51, 52].

Problem 1. To solve the problem (dynamic one-commodity model of Leontiev)

$$b\frac{dx}{dt} = (1-a)x - \omega, \qquad x \ge 0\,, \tag{4.3.1}$$

$$x(0) = x_0, \tag{4.3.2}$$

where $x-$ the amount of gross output produced per unit of time; $a-$ coefficient of production material costs; $b-$ coefficient of incremental capital intensity ratio of the time, $\omega-$ consumption. We use the materials [52] i.e. solution of the problem reduces to the solution of the nonlinear Riccati equation (or rather to the solution of Bernoulli's equation) with a final condition.

If we use the terminology introduced in [51, 52] for the formulation of the problem of control, then the relation (4.3.1)

$$\dot{x} = \frac{1-a}{b} x - \frac{1}{b}\omega . \tag{4.3.3}$$

$$0 \le \omega \le (1-a) x \tag{4.3.4}$$

is the equation of state, $\omega -$ is control, $x -$ is state of the system. Now we need to define a control ω, at restrictions (4.3.2) (4.3.3) (4.3.4). The solution to this problem has to satisfy the condition (4.3.4). Existence of such a decision in this case depends primarily on the choice of discount rate δ and the parameter b. Let's show it. Let us consider the function of Hamilton:

$$H = e^{-\delta t}\left\{ p\left(\frac{1-a}{b} x - \frac{1}{b}\omega\right) + \frac{\alpha}{2}\omega^2 \right\}. \tag{4.3.5}$$

Write a necessary condition for an extremum [93]:

$$\frac{\partial H}{\partial \omega} = e^{-\delta t}\left(-\frac{p}{b} + \alpha\omega\right) = 0. \tag{4.3.6}$$

Hence we have:

$$\omega = \frac{p}{\alpha b}. \tag{4.3.7}$$

The canonical equation for the adjoint variable is written as follows:

$$\frac{d}{dt}\left(e^{-\delta t} p(t)\right) = -\frac{\partial H}{\partial x}. \tag{4.3.8}$$

It follows that

$$\dot{p} = -\left(\frac{1-a}{b} - \delta\right) p. \tag{4.3.9}$$

In view of (4.3.7) rewrite (4.3.3):

$$\dot{x} = \frac{1-a}{b} x - \frac{1}{b^2 \alpha} p. \tag{4.3.10}$$

We will seek $p(t)$ as:

$$p(t) = K(t) x(t) . \tag{4.3.11}$$

Then the expression (4.3.7) is written in the form:

$$\omega = \frac{1}{\alpha b} K x , \tag{4.3.7a}$$

at the time the equation (4.3.10) takes the following form:

$$\dot{x} = \left(\frac{1-a}{b} - \frac{1}{b^2 \alpha} K \right) x . \tag{4.3.12}$$

From the condition of transversality [93, 95] boundary condition $p(t)$ is given by:

$$p(T) = \beta x(T). \tag{4.3.13}$$

From (4.3.11) it follows that

$$p(T) = K(T) x(T). \tag{4.3.14}$$

Comparing the ratio (4.3.8), (4.3.14), we find that

$$K(T) = \beta. \tag{4.3.15}$$

From the equations (4.3.9) and (4.3.12) it is possible to find the differential equation which must be satisfied by the function $K(t)$. Substituting the equation (4.3.9) the expression (4.3.11) and (4.3.12), we obtain:

$$\dot{K} = -\frac{2(1-a)-\delta b}{b} \quad K + \frac{1}{b^2\alpha}K^2 . \tag{4.3.16}$$

Equation (4.3.16) is a Bernoulli's equation. This equation with the boundary condition (4.3.15) determines uniquely the function $K(t)$. Believing $K(t) \neq 0$ and considering the ratio $\frac{1}{K} = V$, $-\frac{1}{K^2}\dot{K} = \left(\frac{1}{K}\right)' = \dot{V}$ from (4.3.16) we get:

$$\dot{V} = \frac{2(1-a)-\delta b}{b}V - \frac{1}{b^2\alpha}, \quad V(T) = \frac{1}{\beta} . \tag{4.3.17}$$

It is a linear equation and solve it, we obtain the solution of Bernoulli's equation:

$$K(t) = \frac{\beta b\alpha(2(1-a)-b\delta)}{e^{(2(1-a)-b\delta)\,(\frac{t-\mathrm{T}}{b})}\left(b\alpha\left(2(1-a)-b\delta\right)-\beta\right)+\beta} . \tag{4.3.18}$$

It should be noted that the solution of the linear equation (4.3.17) is stable, if the following condition:

$$\frac{2(1-a)}{b} - \delta > 0 \text{ at } t < T$$

or

$$\delta < \frac{2(1-a)}{b} . \tag{4.3.19}$$

In view of (4.3.18), (4.3.11) from (4.3.7) we have:

$$\omega(t) = \frac{\beta(2(1-a)-b\delta)x(t)}{e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha\left(2(1-a)-b\delta\right)-\beta\right)+\beta} . \tag{4.3.20}$$

Graphic of function $\varpi(t)$ shown in fig. 4.3.2. We obtain the desired solution, and it have to satisfy the limit (1.1.24). It is easy to notice that in this case the parameter b can not be considered to be small, as it tends to zero condition (4.3.4) is not satisfied. From comparison of (4.3.20) and (4.3.4) we have the inequality:

$$\frac{\beta(2(1-a)-b\delta)}{e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}(b\alpha(2(1-a)-b\delta)-\beta)+\beta} \le 1-a. \qquad (4.3.21)$$

If the parameter b you can not tend to zero then we will need to $T \to \infty$. Then (4.3.21), and $T \to \infty$ we get:

$$\frac{1-a}{b} \le \delta\,. \qquad (4.3.22)$$

Combining (4.3.19) and (4.3.22) we have an interval change values of parameter δ:

$$\frac{1-a}{b} \le \delta < \frac{2(1-a)}{b} \qquad (4.3.23)$$

or

$$1 \le \frac{\delta b}{1-a} < 2\,. \qquad (4.3.24)$$

at $t = T$ we have another condition:

$$\frac{\beta}{b\alpha} \le 1-a\,. \qquad (4.3.25)$$

We show that in the interval $0 \le t \le T$ function $K(t)$ or left side of (4.3.21) is positive definite. The numerator of the fraction is positive. It follows from (4.3.23). It remains to verify the denominator. Let us assume that the denominator - is positive, i.e.

$$e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha(2(1-a)-b\delta)-\beta\right)+\beta>0, \qquad (4.3.26)$$

$$\beta\left(1-e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\right)+e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha(2(1-a)-b\delta)\right)>0.$$

The second term is positive. For the first term was non-negative, it should be

$$1-e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\geq 0$$

or

$$e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\leq 1. \qquad (4.3.27)$$

This inequality holds for all $t\leq T$.

The phase variable $x(t)$, taking into account (4.3.20), is given by:

$$x(t)=e^{\frac{1-a}{b}t}\cdot\frac{(1-a)\left(2-\dfrac{b\delta}{1-a}\right)+\dfrac{\beta}{b\alpha}\left(e^{\frac{1-a}{b}\left(2-\frac{b\delta}{1-a}\right)(T-t)}-1\right)}{(1-a)\left(2-\dfrac{b\delta}{1-a}\right)+\dfrac{\beta}{b\alpha}\left(e^{\frac{1-a}{b}\left(2-\frac{b\delta}{1-a}\right)T}-1\right)}x_0\,. \qquad (4.3.28)$$

On the fig. 4.3.1 is a charting function.

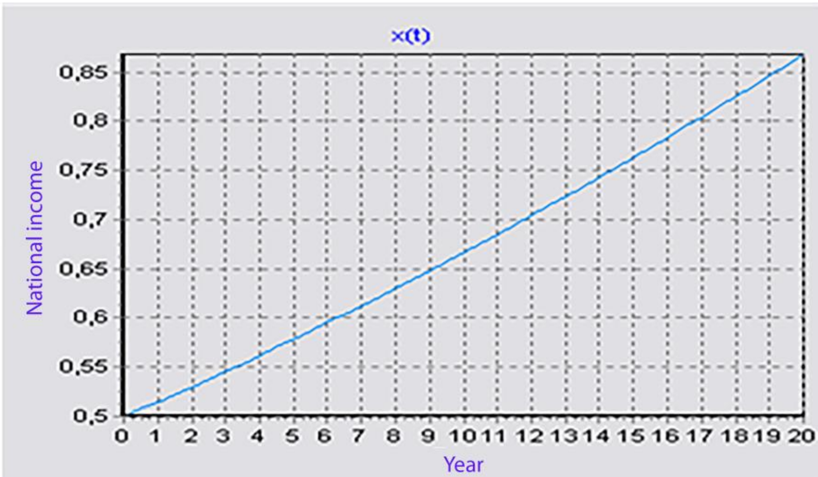

Fig. 4.3.1 *Schedule of function* $x(t)$ *(formula 4.3.28).*

So, we have indicated the conditions (4.3.27), under which there is a solution to this problem in the form (4.3.28). On the fig. 4.3.3 is shown the optimal regulator of system.

"Amplification coefficient" $K(t)$ obtained by modeling equation (4.3.17).

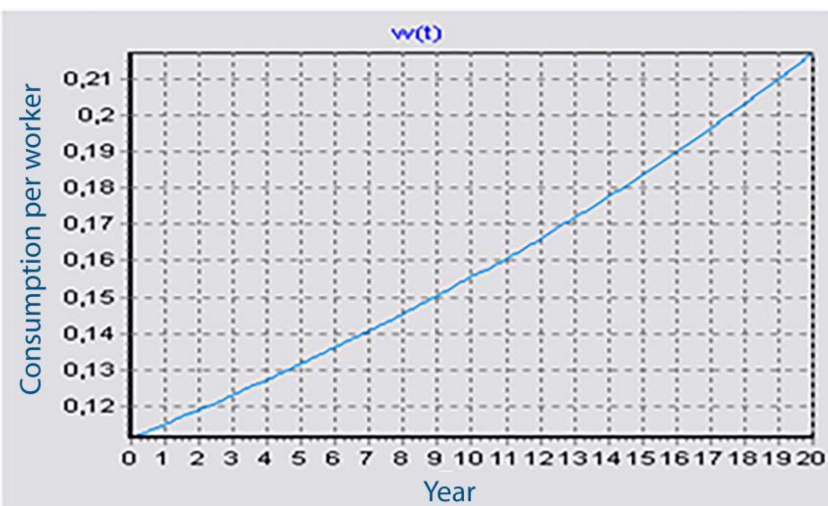

Fig. 4.3.2 *Schedule of function* $\omega(t)$ *(formula 4.3.20).*

Problem 2**.** Now consider another task. In this case the quality of the process is estimated linear functional:

$$J = \int_0^T e^{-\delta t} w(t)dt \to \max \tag{4.3.29}$$

or

$$J = -\int_0^T e^{-\delta t} w(t)dt \to \min \; . \tag{4.3.29}$$

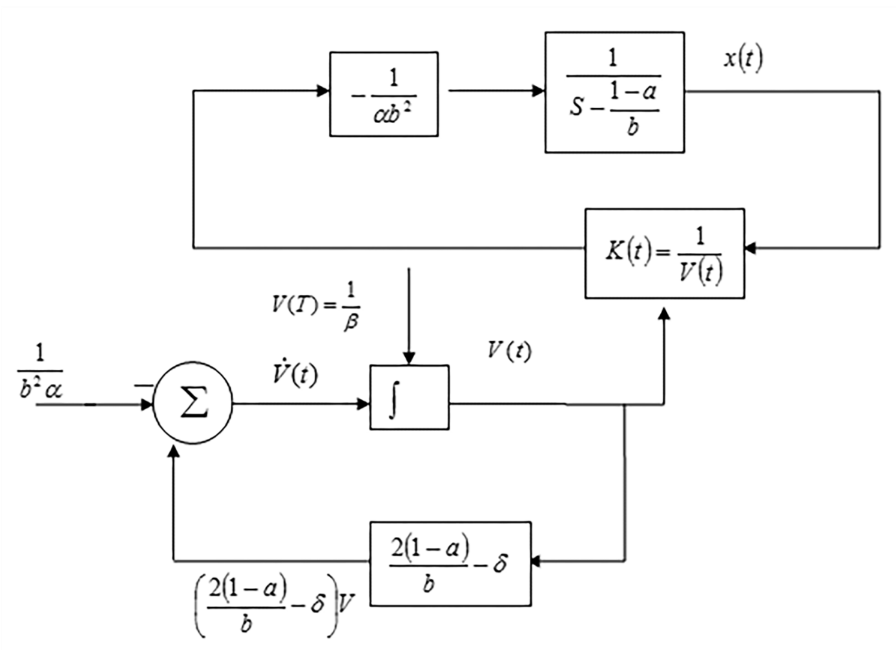

Fig. 4.3.3 *Optimum regulator of system* $\dot{x}(t) = \frac{1-a}{b}x(t) - \frac{1}{b}\omega(t)$.

We also assume the intensity of consumption $w(t)$ can not exceed a certain maximum level w^*, so that

$$0 \le w \le w^* \; . \tag{4.3.30}$$

For solve this problem we use the method set out in [41]. Considering the consequent economic sense variables (they are not negative), we can not take the type of restriction

$$\max_{t_0 \le t \le T} |u(t)| \le \nu. \tag{4.3.31}$$

Therefore, we need to introduce a new function to get artificially limiting the type (4.3.31) from (4.3.30). So, enter the following function:

$$u = w - \frac{w^*}{2}. \tag{4.3.32}$$

Then in view of (4.3.32) from (4.3.30), we obtain:

$$-\frac{w^*}{2} \le u \le \frac{w^*}{2} \tag{4.3.33}$$

or

$$|u(t)| \le \frac{w^*}{2}. \tag{4.3.34}$$

Now we need to change the dynamic, respectively, one-commodity model Leontiev, which represents the balance ratio [86]

$$b\frac{dx}{dt} = (1-a)x - \omega, \qquad x \ge 0, \tag{4.3.35}$$

$$x(0) = x_0, \tag{4.3.36}$$

where $x-$ the amount of gross output produced per unit of time; $a-$ coefficient of production material costs; $b-$ coefficient of incremental capital intensity ratio of the time, $\omega-$ consumption:

$$b\frac{dx}{dt} = (1-a)x - u - \frac{w^*}{2}, \quad x \ge 0, \tag{4.3.37}$$

$$x(0) = x_0 \,. \tag{4.3.38}$$

Horizon process time distribution of gross product is considered as final. Then it is necessary to set the final time the minimum acceptable value of the intensity of the gross output in order to allow consumption and outside this time horizon:

$$x(T) = x_T \,. \tag{4.3.39}$$

Then we need to solve the following problem: find a control $u(t)$, minimizes the function (4.3.29) with restrictions (4.3.34) - (4.3.39).

This problem reduces to the problem of moments. Solution of the equation (4.3.37) with the initial condition (4.3.38) can be written in the form:

$$x(t) = e^{\left(\frac{1-a}{b}\right)t} x_0 - \frac{1}{b}\int_0^T e^{\left(\frac{1-a}{b}\right)(t-s)} u(s)\,ds + \frac{w^*}{2(1-a)}\left(1 - e^{\left(\frac{1-a}{b}\right)t}\right). \tag{4.3.40}$$

Hence, when at $t = T$ we will have:

$$\alpha = \int_0^T e^{-\left(\frac{1-a}{b}\right)t} u(t)\,dt, \tag{4.3.41}$$

where $\alpha = b\left(-x_T + e^{\left(\frac{1-a}{b}\right)T} x_0 + \frac{w^*}{2(1-a)}\left(1 - e^{\left(\frac{1-a}{b}\right)T}\right)\right)$.

Therefore, we need to find a control $u(t)$, which minimizes the functional (4.3.29) with restrictions (4.3.34), (4.3.41).

According to [51] ($u^0(t) = \nu sign\left\{ B_0'(t)\Phi'(T,t) l_1^{(0)} + B_2'(T) e^{-A_4'(t)\left(\frac{T-t}{\mu}\right)} l_2^{(0)} - e^{-\delta t} \right\}$),

the control is written in the form:

$$u(t) = \frac{w^*}{2} sign\left(\lambda_1 e^{-\left(\frac{1-a}{b}\right)(t-T)} - e^{-\delta t}\right)., \tag{4.3.42}$$

function $\lambda_1 e^{-\left(\frac{1-a}{b}\right)(t-T)} - e^{-\delta t}$ can change sign more than once.

Therefore, setting $\lambda_1 e^{-\left(\frac{1-a}{b}\right)(s-T)} - e^{-\delta s} = 0,$ we find:

$$s = \frac{1}{\frac{1-a}{b} - \delta}\left(\ln \lambda_1 + \frac{1-a}{b}T\right). \tag{4.3.43}$$

Believing $s \in (0,T)$ and substituting (3.6.42) into (4.3.41), we obtain the following relations:

$$s = \frac{1}{\frac{1-a}{b} - \delta}\left(\ln \lambda_1 + \frac{1-a}{b}T\right), \tag{4.3.44}$$

$$\lambda_1 = e^{-\frac{1-a}{b}T}\left\{\frac{1}{2}\left(1 + e^{-\left(\frac{1-a}{b}\right)T}\right) - \frac{\alpha(1-a)}{w^* b} e^{-\left(\frac{1-a}{b}\right)T}\right\}^{\frac{b\delta}{1-a}-1}. \tag{4.3.45}$$

According to [51], control (4.3.42) have to satisfy the following moment ratio

$$\int_0^T e^{-\delta t} u(t)\, dt = \alpha\lambda_1 - 1. \tag{4.3.46}$$

Substituting (4.3.45) to (4.3.43), we obtain:

$$s = -T - \frac{b}{1-a-b\delta}\ln\left(\frac{1}{2}(e^{(\frac{1-a}{b})T} + 1) - \frac{\alpha(1-a)}{w^* b}\right), \tag{4.3.47}$$

in order to $s \in (0,T)$ have to have the following conditions:

$$0 < \frac{b\delta}{1-a} < 1, \tag{4.3.48}$$

$$\frac{1}{2}\left(e^{(\frac{1-a}{b}T)} + 1\right) - \frac{\alpha(1-a)}{w^* b} > 1. \tag{4.3.49}$$

From (4.3.49), we have:

$$|\alpha| < \frac{w^* b}{2(1-a)}\left(e^{\frac{1-a}{b}T} - 1\right). \tag{4.3.50}$$

So we have a necessary condition for the existence of control $u(t)$, that satisfies the relations of moment (4.3.41), (4.3.46) and limitation (4.3.34).

Now substituting control (4.3.42) to the equation (4.3.46) and taking into account the relations (4.3.44), (4.3.45), we obtain the following equation for the parameter λ_1:

$$2e^{\frac{\delta T}{1-\frac{b\delta}{1-\alpha}}}\left(1-\frac{b\delta}{1-a}\right) - \left(\frac{2\delta}{w^*} + e^{-\delta T} - 1\right)\lambda_1^{\frac{\frac{b\delta}{1-a}}{1-\frac{b\delta}{1-a}}} + \frac{b\delta}{1-a}\left(e^{(\frac{1-a}{b})T} + 1\right)\lambda_1^{\frac{1}{1-\frac{b\delta}{1-a}}} = 0. \tag{4.3.51}$$

We introduce the notation $\mu = \frac{b\delta}{1-a}$. Then the equation (4.3.51) can be written as:

$$\mu\left(e^{\frac{1-a}{b}T} + 1\right)\lambda_1^{\frac{1}{1-\mu}} - \left(\frac{2\delta}{w^*} + e^{-\delta T} + 1\right)\lambda_1^{\frac{\mu}{1-\mu}} + 2e^{\frac{\delta T}{1-\mu}}(1-\mu) = 0. \tag{4.3.52}$$

For different values μ satisfying (4.3.48) we have the equation of a different nature. So at $\mu \neq 1$ and natural $n = \frac{\mu}{1-\mu}$ available equation of at least second order. Believing $n = \frac{\mu}{1-\mu} = 1$, we find $\mu = \frac{1}{2}$. This value satisfies the condition

(4.3.48). Then, to determine the parameters λ_1 we obtain the following quadratic equation

$$\left(e^{\left(\frac{1-a}{b}\right)T}+1\right)\lambda_1^2-2\left(\frac{2\delta}{w^*}+e^{-\delta T}+1\right)\lambda_1+2e^{2\delta T}=0. \qquad (4.3.53)$$

This equation has real solutions, if the condition

$$\left(\frac{2\delta}{w^*}+e^{-\delta T}+1\right)^2-2e^{2\delta T}\left(e^{\left(\frac{1-a}{b}\right)T}+1\right)\geq 0. \qquad (4.3.54)$$

For to select $b,\ \ \delta$ we have the following relationship:

$$\delta=\frac{1-a}{2b} \text{ or}$$

$$b=\frac{1-a}{2\delta}. \qquad (4.3.55)$$

The positive solution of the equation (4.3.53) is the desired parameter optimal control (4.3.42). After determining the desired value $\lambda_1=\lambda_1^*$, solution of the this problem can be written as:

$$w=\frac{w^*}{2}+\frac{w^*}{2}sign\left(\lambda_1^*e^{-\left(\frac{1-a}{b}\right)(t-T)}-e^{-\delta t}\right). \qquad (4.3.56)$$

Consumption w defined in the form (3.6.56) satisfies the restriction (4.3.30). Then under consideration on the interval will have the following path:

$$x(t)=\begin{cases} e^{(\frac{1-a}{b})t}\left(x_0-\dfrac{w^*}{1-a}\right)+\dfrac{w^*}{1-a}, & 0\le t<s,\\[2ex] \dfrac{\dfrac{w^*}{1-a}\,x_T}{x_T-e^{(\frac{1-a}{b})T}\left(x_0-\dfrac{w^*}{1-a}\right)}, & t=s,\\[2ex] e^{(\frac{1-a}{b})(t-T)}\cdot x_T, & s<t\le T,\end{cases}$$

where $$s=T-\frac{2b}{1-a}\ln\frac{1-a}{w^*}\left(\cancel{x_T}-e^{(\frac{1-a}{b})T}x_0-\frac{w^*}{1-a}\right).$$

The graph of this function and the table are shown in fig. 4.3.4.

It should be noted that in this case the process of distribution of gross domestic product is characterized by a singularly perturbed equation, since the (4.3.35) can be written as

$$\mu\frac{dx}{dt}=\delta x-\frac{\delta\omega}{1-a}, \tag{4.3.57}$$

where $\mu=\dfrac{b\delta}{1-a}<1-$ small parameter.

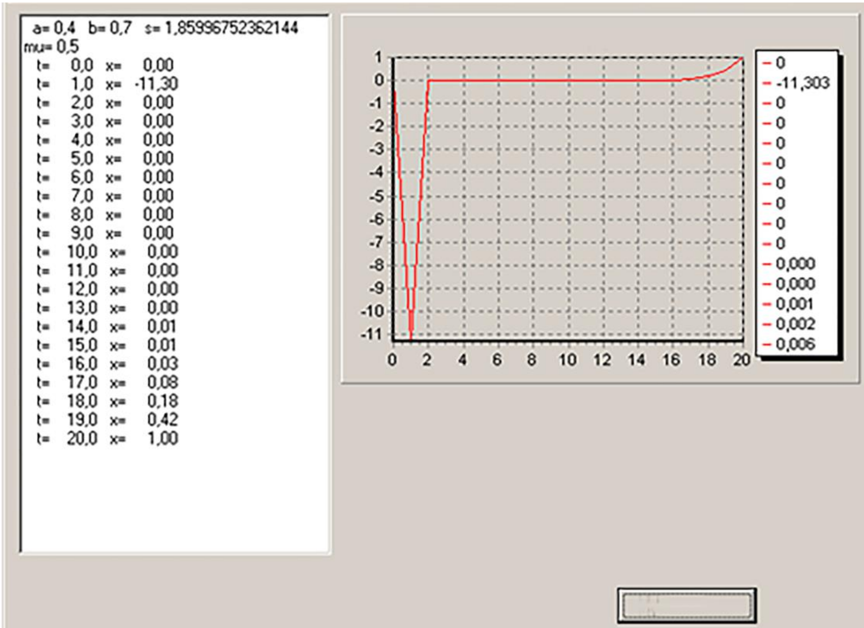

Fig. 4.3.4 *Graph of function* $x(t)$ *and table of values.*

4.4 Investigation of the Problem on Optimal Control in Single-Commodity Macro Models Based on the Delay of the Process of Investments

As decomposition described in chapter 1, for the task

$$\dot{k} = -(\varepsilon + n)k + \upsilon, \tag{4.4.1}$$

$$\mu\dot{\upsilon} = (1-a)\,fk - (1+\mu n)\upsilon - \bar{w},$$

where $\mu = \dfrac{1}{\lambda}$, $\quad \bar{w} = w - (1-a)r$ we will share the slow and fast position.

Matrix A and B have the form:

$$A = \begin{pmatrix} -(\varepsilon+n) & 1 \\ \dfrac{(1-a)\,f}{\mu} & -\dfrac{1+\mu n}{\mu} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In this case

$$A_1 = -(\varepsilon + n), \qquad A_2 = 1, \qquad A_3 = (1-a) f,$$
$$A_4 = -(1+\mu n), \qquad B_1 = 0, \qquad B_2 = -1.$$

Then the system with separated variables can be written as:

$$\begin{aligned} \dot{\tilde{k}}_\mu &= m_1 \tilde{k}_\mu - N\bar{w}_\mu, \\ \mu \dot{\tilde{\upsilon}}_\mu &= m_2 \tilde{\upsilon}_\mu - \bar{w}_\mu, \end{aligned} \tag{4.4.2}$$

where

$$m_1 = -(\varepsilon + n) + \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2\mu},$$
$$m_2 = -(1+\mu n) - \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2}$$

$$N = \frac{1}{\sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}, \;\; \tilde{\upsilon}_\mu = \upsilon_\mu - H_1 k_\mu, \quad \tilde{k}_\mu = k_\mu + \mu N \tilde{\upsilon}_\mu,$$

$$\bar{w}_\mu = w_\mu - (1-a) r, \quad w_\mu(t) = w(t,\mu), \quad \tilde{\upsilon}_\mu(t) = \tilde{\upsilon}(t,\mu), \quad \tilde{k}_\mu(t) = \tilde{k}(t,\mu).$$

For the system (4.4.2), we have the following boundary conditions:

$$\tilde{k}_\mu(0) = \tilde{k}_0, \qquad \tilde{\upsilon}_\mu(0) = \tilde{\upsilon}_0, \tag{4.4.3}$$

$$\tilde{k}_\mu(T) = \tilde{k}_T, \qquad \tilde{\upsilon}_\mu(T) = \tilde{\upsilon}_T, \tag{4.4.4}$$

where $\begin{aligned} \tilde{k}_0 &= k_0 + \mu N \tilde{\upsilon}_0, & \tilde{\upsilon}_0 &= \upsilon_0 - H_1 k_0, \\ \tilde{k}_T &= k_T + \mu N \tilde{\upsilon}_T, & \tilde{\upsilon}_T &= \upsilon_T - H_1 k_T, \end{aligned}$

$$H_1 = \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2\mu}, \; \lim_{\mu \to 0} H_1(\mu) = (1-a) f.$$

It is easy can to make sure that inequality

$$\left(1-a\right)f<\left(\varepsilon+n\right)\left(1+\mu n\right) \tag{4.4.5}$$

is a condition of the sustainability systems (4.4.2).

For the system (4.4.1) with the boundary conditions (4.4.3), (4.4.4) consider the problem of choosing a trajectory of consumption per worker at the condition

$$J=-\int_0^T e^{-\delta s}w_{\varpi}\left(s\right)ds\to\min, \tag{4.4.6}$$

or $J=\int_0^T e^{-\delta s}w_{\varpi}\left(s\right)ds\to\max.$

In the future, we believe that for the system (4.4.2) run the condition (4.4.5), i.e. the system is stable. In this problem, the phase coordinates are the basic production assets for the one worker k and volume of investments was put into effect, calculated per worker υ and consumption per worker w - is a control parameter. It should be noted that the consumption values can not less than zero and in the closed economy can not rise higher output per worker, i.e.

$$0\le w\le fk_*+r\,, \tag{4.4.7}$$

where k_*- the maximum allowable level of capital per worker.

Solution of the system (4.4.2) with the initial conditions (4.4.3) can be written as:

$$\tilde{k}_{\mu}(t)=e^{m_1 t}\tilde{k}_0-\int_0^t e^{m_1(t-s)}N\overline{w}_{\mu}(s)ds,$$

$$\tilde{\upsilon}_{\mu}(t)=e^{m_2\frac{t}{\mu}}\tilde{\upsilon}_0-\frac{1}{\mu}\int_0^t e^{m_2\frac{(t-s)}{\mu}}\overline{w}_{\mu}(s)ds. \tag{4.4.8}$$

At $t = T$, taking into account the final conditions (4.4.4) from (4.4.8) we have the following relation of moment:

$$c_1 = \int_0^T e^{m_1(T-s)} N\bar{w}_\mu(s)ds, \quad c_2 = \frac{1}{\mu}\int_0^T e^{m_2(\frac{T-s}{\mu})}\bar{w}_\mu(s)ds, \tag{4.4.9}$$

where $c_1 = -\tilde{k}_T + e^{m_1 T}\tilde{k}_0, \quad c_2 = -\tilde{\upsilon}_T + e^{m_2\frac{T}{\mu}}\tilde{\upsilon}_0.$

Considering that $\bar{w}_\mu = w_\mu - (1-a)r$ and from (4.4.6) we have:

$$J = \int_0^T e^{-\delta s}\bar{w}_\mu(s)ds + \int_0^T e^{-\delta s}(1-a)rds\,. \tag{4.4.10}$$

The second integral does not depend on $\bar{w}_\mu$ and the desired minimum of $\bar{w}_\mu$ value $J(\bar{w}_\mu)$ will be reached on the same functions $\bar{w}_\mu = \bar{w}_{\mu_{onm}}(t)$, and that the minimum of expression

$$J_1 = \int_0^T e^{-\delta s}\bar{w}_\mu(s)ds. \tag{4.4.11}$$

From (4.4.7) we have the following restrictions for function $\bar{w}_\mu$:

$$0 \le \bar{w}_\mu \le fk_* + r. \tag{4.4.12}$$

Then

$$0 \le \int_0^T \bar{w}_\mu^2(t)dt \le l, \tag{4.4.13}$$

where $l = (fk_* + r)^2 T.$

Thus, have a problem about the minimum of (4.4.11) with restrictions (4.4.9), (4.4.13).

A similar task can be solved by the method of A. I. Egorov [37], which is based on theorem of Levi about orthogonal decomposition of elements of a Hilbert space. The procedure for constructing an optimal control method of A. I. Egorov differs little from of the same scheme as in the method of moments [53]. But it is convenient for the practical construction of optimal control, it is not required to solve the auxiliary extreme problems. Desired control take in the form:

$$\bar{w}_\mu\left(t\right)=\gamma_1 e^{m_1(T-t)}N+\gamma_2 e^{m_2\left(\frac{T-t}{\mu}\right)}+\gamma_0 e^{-\delta t}\,. \tag{4.4.14}$$

This control belongs to a boundary of the set (4.4.13) [96]:

$$\int_0^T \bar{w}_\mu^2\left(t\right)dt=l. \tag{4.4.15}$$

Constants $\gamma_1,\gamma_2,\gamma_0$ will be determined from the relation (4.4.9), (4.4.15). Substituting the value of $\bar{w}(t)$ from formula (4.4.14) to the relation (4.4.9), (4.4.15), we obtain a system of algebraic equations

$$r_{11}\gamma_1+r_{12}\gamma_2+r_{13}\gamma_0=c_1,\; r_{21}\gamma_1+r_{22}\gamma_2+r_{23}\gamma_0=c_2, \tag{4.4.16}$$

$$r_{11}\gamma_1^2+r_{22}\gamma_2^2+r_{33}\gamma_0^2+2r_{12}\gamma_1\gamma_2+2r_{13}\gamma_1\gamma_0+2r_{23}\gamma_2\gamma_0=l,$$

where $r_{11}=\dfrac{N^2}{2m_1}(e^{2m_1T}-1),\quad r_{12}=\dfrac{N\mu}{\mu m_1+m_2}(e^{(m_1\mu+m_2)\frac{T}{\mu}}-1),\quad r_{13}=\dfrac{e^{m_1T}N}{m_1+\delta}(1-e^{-(m_1+\delta)T}),$

$$\mu r_{21}=r_{12},\;\; r_{22}=\frac{1}{2m_2}(e^{\frac{2m_2T}{\mu}}-1),\;\; r_{23}=\frac{e^{\frac{m_2T}{\mu}}}{m_2+\mu\delta}(1-e^{-\frac{m_2+\delta\mu}{\mu}T}),\;\; r_{33}=\frac{1}{2\delta}(1-e^{-2\delta T}).$$

Of the first two equations of the systems (4.4.15) γ_1,γ_2 are uniquely determined through γ_0:

$$\gamma_1 = d_1\gamma_0 + h_1,\ \gamma_2 = d_2\gamma_0 + h_2, \tag{4.4.17}$$

where $d_1 = \dfrac{r_{12}r_{23} - r_{13}r_{22}}{r_{11}r_{22} - r_{21}r_{12}}$, $h_1 = \dfrac{c_1 r_{22} - c_2 r_{12}}{r_{11}r_{22} - r_{12}r_{21}}$, $d_2 = \dfrac{r_{12}r_{13} - r_{11}r_{23}}{r_{11}r_{22} - r_{12}r_{21}}$,

$h_2 = \dfrac{c_2 r_{11} - c_1 r_{12}}{r_{11}r_{22} - r_{12}r_{21}}$, $r_{11}r_{22} - r_{12}r_{21} \neq 0.$

Substituting the value of γ_1 and the last equation of the system (4.4.16), get the quadratic equation with respect γ_0:

$$a\gamma_0^2 + 2b\gamma_0 + c = 0, \tag{4.4.18}$$

where $a = r_{11}d_1^2 + r_{22}d_2^2 + r_{33} + 2r_{12}d_1d_2 + 2r_{13}d_1 + 2r_{23}d_2$,

$$b = r_{11}d_1h_1 + r_{22}d_2h_2 + r_{12}d_1h_2 + r_{12}d_2h_1 + r_{13}h_1 + r_{23}h_2,$$
$$c = r_{11}h_1^2 + r_{22}h_2^2 + 2r_{12}h_1h_2 - l.$$

Equation (4.4.18) has two real roots $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, if they performed condition

$$b^2 - ac \geq 0. \tag{4.4.19}$$

Substituting the found values γ_0 in (4.4.17), obtain:

$$\gamma_1^{(1)} = d_1\gamma_0^{(1)} + h_1,\ \gamma_1^{(2)} = d_1\gamma_0^{(2)} + h_1,\ \gamma_2^{(1)} = d_2\gamma_0^{(1)} + h_2,$$
$$\gamma_2^{(2)} = d_2\gamma_0^{(2)} + h_2. \tag{4.4.20}$$

Then by the expression (4.4.14) we define two functions

$$\overline{w}_\mu^{(1)}(t) = \gamma_1^{(1)} e^{m_1(T-t)} N + \gamma_2^{(1)} e^{m_2(\frac{T-t}{\mu})} + \gamma_0^{(1)} e^{-\delta t},$$

$$\overline{w}_\mu^{(2)}(t) = \gamma_1^{(2)} e^{m_1(T-t)} N + \gamma_2^{(2)} e^{m_2(\frac{T-t}{\mu})} + \gamma_0^{(2)} e^{-\delta t}. \tag{4.4.21}$$

One of them is the solution of the considered task. Everything, from these controls by substituting in functional (4.4.11), we will define the optimum decision which corresponds to the minimum value of functional. We write down optimal control as

$$\overline{w}_{\mu_{onm}}(t) = \gamma_1^{(\mu)} e^{m_1(T-t)} N + \gamma_2^{(\mu)} e^{m_2 \frac{(T-t)}{\mu}} + \gamma_0^{(\mu)} e^{-\delta t}, \tag{4.4.22}$$

where $\gamma_0^{(\mu)}, \gamma_1^{(\mu)}, \gamma_2^{(\mu)} -$ optimum coefficients which depend on small parameter. Then the phase coordinates corresponding to optimum control (4.4.22) are defined by ratios:

$$\begin{aligned} k_{\mu_{onm}}(t) &= e^{m_1 t}\tilde{k}_0 - \mu N \tilde{\upsilon}_0 + \frac{N^2 e^{m_1 T}}{2m_1}\left(e^{-m_1 t} - e^{m_1 t}\right)\gamma_1^{(\mu)} + \frac{e^{\frac{m_2 T}{\mu}} N\mu}{\mu m_1 + m_2}\left(e^{-\frac{m_2 t}{\mu}} - e^{m_1 t}\right)\gamma_2^{(\mu)} + \\ &+ \frac{N}{m_1 + \delta}\left(e^{-\delta t} - e^{m_1 t}\right)\gamma_0^{(\mu)}, \end{aligned} \tag{4.4.23}$$

$$\begin{aligned} \upsilon_{\mu_{onm}}(t) &= e^{\frac{m_2 t}{\mu}}\tilde{\upsilon}_0 - Hk_{\mu_{onm}}(t) + \frac{Ne^{m_1 T}}{m_1\mu + m_2}(e^{-m_1 t} - e^{\frac{m_2 t}{\mu}})\gamma_1^{(\mu)} - \frac{e^{\frac{m_2(T+t)}{\mu}}}{2m_2}(1 - e^{-\frac{2m_2 t}{\mu}})\gamma_2^{(\mu)} + \\ &\frac{1}{\mu\delta + m_2}(e^{-\delta t} - e^{\frac{m_2 t}{\mu}})\gamma_0^{(\mu)}. \end{aligned} \tag{4.4.24}$$

On fig. 4.4.1 are graphs of control $\overline{w}_{\mu_{onm}}(t)$ and functions $\tilde{k}_{\mu_{onm}}(t)$, $\tilde{\upsilon}_{\mu_{onm}}(t)$ respectively at $\mu = 0{,}5$. At $\mu \to 0$ we have the following limit ratios:

$$m_1 \to -(\varepsilon + n) + (1-a)f = m_0, \ \ m_2 \to -1,$$

$$N \to 1, \ H_1 \to (1-a)f, \ r_{11} \to r_{11}^{(0)} = \frac{1}{2m_0}(e^{2m_0 T} - 1),$$

$$r_{12} \to r_{12}^{(0)} = 0, \quad r_{21} \to r_{21}^{(0)} = 0,$$

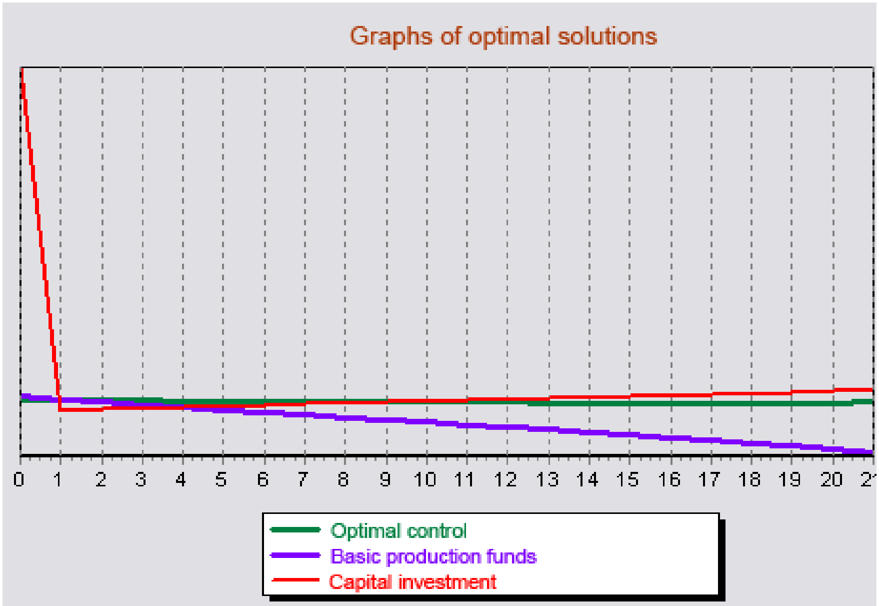

Fig. 4.4.1 *Graph of optimum decisions.*

$$r_{13} \to r_{13}^{(0)} = \frac{e^{m_0 T}}{m_0 + \delta}(1 - e^{-(m_0+\delta)T}),\, r_{22} \to r_{22}^{(0)} = \frac{1}{2},\, r_{23} \to r_{23}^{(0)} = e^{-\delta T},$$

$$r_{33} = r_{33}^{(0)} = \frac{1}{2\delta}(1 - e^{-2\delta T}),$$

$$d_1 \to d_1^{(0)} = -\frac{r_{13}^{(0)}}{r_{11}^{(0)}},\, d_2 \to d_2^{(0)} = -2r_{23}^{(0)},\; c_1 \to c_1^{(0)} = -k_T + e^{m_0 T} k_0, \quad (4.4.25)$$

$$c_2 \to c_2^{(0)} = -\upsilon_T + (1-a)\, fk_T,\; h_1 \to h_1^{(0)} = \frac{c_1^{(0)}}{r_{11}^{(0)}},\; h_2 \to h_2^{(0)} = 2c_2^{(0)},\; r_{11}^{(0)} \neq 0.$$

For limit values of parameters from (4.4.25) it is possible to receive the ratios similar (4.4.18) - (4.4.21) and to define from them limit values of optimum coefficients: $\gamma_0^{(\mu)} \to \gamma_0^{(0)}$, $\gamma_1^{(\mu)} \to \gamma_1^{(0)}$, $\gamma_2^{(\mu)} \to \gamma_2^{(0)}$. Then from (4.4.22), (4.4.23) and (4.4.24) at $\mu \to 0$, because of the system (4.4.2) stability we receive the following limit ratios of the solution of a task respectively (see fig. 4.4.2)

$$w^{(0)}(t) = e^{m_0(T-t)}\gamma_1^{(0)} + e^{-\delta t}\gamma_0^{(0)} + (1-a)r\,, \qquad (4.4.26)$$

$$k^{(0)}(t) = e^{m_0 t}k_0 + \frac{e^{m_0 T}}{2m_0}(e^{-m_0 t} - e^{m_0 t})\gamma_1^{(0)} + \frac{1}{m_0+\delta}(e^{-\delta t} - e^{m_0 t})\gamma_0^{(0)}, \quad (4.4.27)$$

$$\upsilon^{(0)}(t) = k^{(0)}(t)\cdot(1-a)f - \overline{w}^{(0)}(t) = (1-a)(fk^{(0)}(t)+r) - w^{(0)}(t). \quad (4.4.28)$$

We will notice that direct application the method’s of A. I. Egorov [37] to this task increases number of the equations by one. If the studied system has high dimension, the method will be ineffective. Therefore we pass on another.

We will consider system

$$\dot{k}_\mu = m_0 k_\mu - \overline{w}_\mu^*,\;\; k_\mu(0) = k_0\,,\;\; k_\mu(T) = k_T\,, \qquad (4.4.29)$$

$$\mu\dot{\upsilon}_\mu^* = -\upsilon_\mu^* - \overline{w}_\mu^*,\;\; \upsilon_\mu^*(0) = \upsilon_0 - (1-a)f\kappa_0 = \upsilon_0^*,\;\; \upsilon_\mu^*(T) = \upsilon_{\mathrm{T}} - (1-a)fk_T = \upsilon_T^*,$$

where $\upsilon_\mu^* = \upsilon_\mu - (1-a)fk_\mu$.

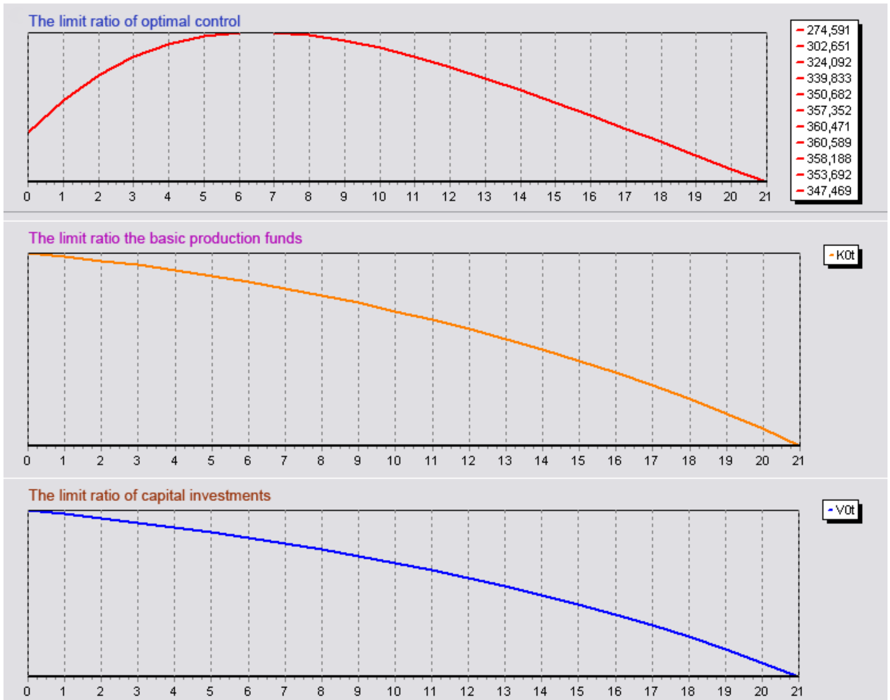

Fig. 4.4 *Limit ratios.*

Now we will consider a task about a minimum of functional (4.4.11) at restrictions (4.4.29) and (4.4.13). We will designate this task a symbol P_μ. At $\mu = 0$ from (4.4.29) we receive the generating system

$$\dot{k}^{(0)} = m_0 k^{(0)} - \overline{w}^{(0)}, \quad k^0(0) = k_0, \quad k^{(0)}(T) = k_T, \qquad (4.4.30)$$

$$0 = -\upsilon^* - \overline{w}^{(0)} \text{ or } \upsilon^{(0)} = (1-a)\, fk^{(0)} - \overline{w}^{(0)},$$

where $\upsilon^* = \upsilon^{(0)} - (1-a)\, fk^{(0)}$.

Then tasks (4.4.11), (4.4.13), (4.4.30) are limit in relation to a task P_μ. We will designate it through P_0.

We solve the task P_0. We write down the decision of system (4.4.30) in a look:

$$k^0(t) = e^{m_0 t} k_0 - \int_0^t e^{m_0(t-s)} \overline{w}^{(0)}(s) ds, \tag{4.4.31}$$

$$\upsilon^{(0)}(t) = (1-a) f k^{(0)}(t) - \overline{w}^{(0)}(t).$$

At $t = T$ from the first equation (4.4.31), we obtain:

$$c_1^{(0)} = \int_0^T e^{m_0(T-s)} \overline{w}^{(0)}(s) ds, \tag{4.4.32}$$

where $c_1^{(0)} = -k_T + e^{m_0 T} k_0$.

Similarly, above stated method, control $\overline{w}^{(0)}(t)$ will be search in the form of:

$$\overline{w}^{(0)}(t) = \overline{\gamma}_1 e^{m_o(T-t)} + \overline{\gamma}_0 e^{-\delta t}. \tag{4.4.33}$$

This control belongs to the boundary of the set (4.4.13), i.e. it must satisfy the equation

$$\int_0^T \overline{w}^{(0)^2}(t) dt = l. \tag{4.4.34}$$

Given the (4.6.33) by (4.4.32), (4.6.34), we obtain the following system of algebraic equations $\overline{\gamma}_1, \overline{\gamma}_0$:

$$\begin{cases} r_{11}^{(0)} \overline{\gamma}_1 + r_{13}^{(0)} \overline{\gamma}_0 = c_1^{(0)}, \\ r_{11}^{(0)} \overline{\gamma}_1^2 + 2 r_{13}^{(0)} \overline{\gamma}_1 \overline{\gamma}_0 + r_{33}^{(0)} \overline{\gamma}_0^2 = l, \end{cases} \tag{4.4.35}$$

where $r_{11}^{(0)}$, $r_{13}^{(0)}$, $r_{33}^{(0)}$ defined by the relations of formula (4.4.25). From the first equation (4.6.35), we obtain:

$$\bar{\gamma}_1 = \bar{d}_1 \bar{\gamma}_0 + \bar{h}_1, \tag{4.4.36}$$

where $\bar{d}_1 = -\dfrac{r_{13}^{(0)}}{r_{11}^{(0)}}, \quad \bar{h}_1 = \dfrac{c_1^{(0)}}{r_{11}^{(0)}}$. Given the (4.6.36), from the second equation (4.6.35), we obtain a quadratic equation for $\bar{\gamma}_0$:

$$\bar{a}\bar{\gamma}_0^2 + 2\bar{b}\,\bar{\gamma}_0 + \bar{c} = 0, \tag{4.4.37}$$

where $\bar{a} = r_{11}^{(0)}\bar{d}_1^2 + 2r_{13}^{(0)}\bar{d}_1 + r_{33}^{(0)}, \quad \bar{b} = r_{11}^{(0)}\bar{d}_1\bar{h}_1 + r_{13}^{(0)}\bar{h}_1, \quad \bar{c} = r_{11}^{(0)}\bar{h}_1^2 - l.$

Equation (4.6.37) has two real roots $\bar{\gamma}_0^{(1)}, \quad \bar{\gamma}_0^2$, if $\bar{b}^2 - \bar{a}\bar{c} \geq 0$. Assume that these roots exist. From (4.6.33) we have two functions -control, one of which minimizes the functional (4.4.11), written in the form (see. fig.4.4.3)

$$w^{(0)}(t) = \bar{\gamma}_1^{(0)} e^{m_0(T-t)} + \bar{\gamma}_0^{(0)} e^{-\delta t} + (1-a)r, \tag{4.4.38}$$

where $\bar{\gamma}_1^{(0)}, \quad \bar{\gamma}_0^{(0)}$ – optimal coefficients. Substituting (4.4.38) to (4.4.31), we obtain (fig. 4.4.3):

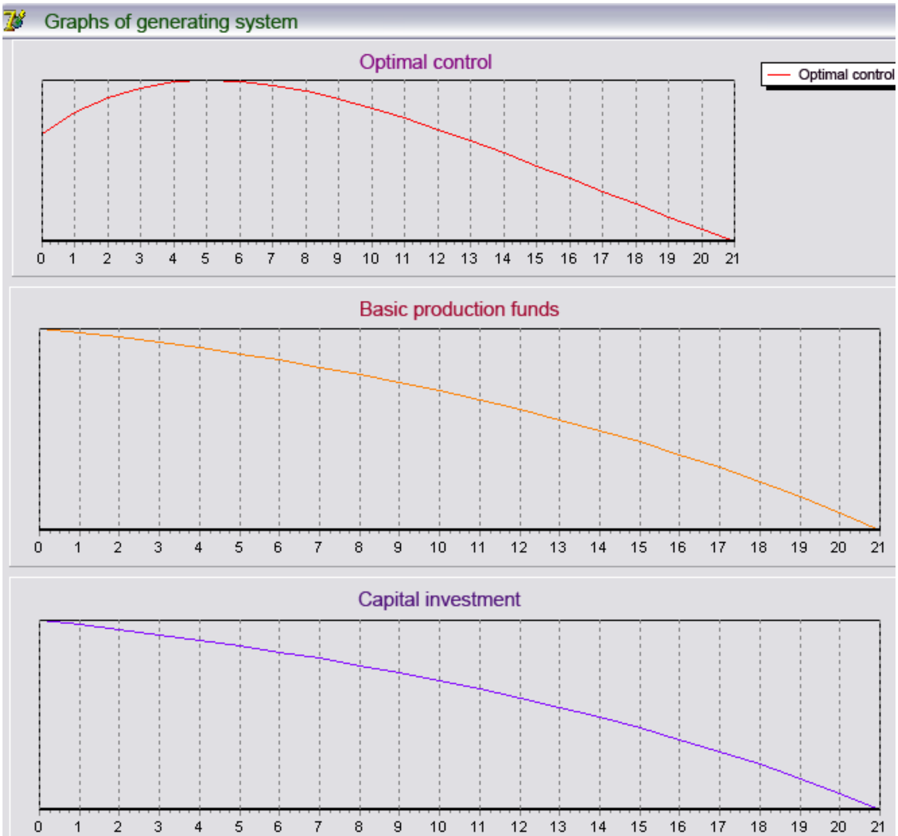

Fig. 4.4.3 *The graphs generating systems.*

$$k^{(0)}(t) = e^{m_0 t} k_0 + \frac{e^{m_0 T}}{2m_0}(e^{-m_0 t} - e^{m_0 t})\overline{\gamma}_1^{(0)} + \frac{1}{m_0 + \delta}(e^{-\delta t} - e^{m_0 t})\overline{\gamma}_0^{(0)}, \tag{4.4.39}$$

$$\upsilon^{(0)}(t) = -\overline{w}^{(0)}(t) + (1-a) f \cdot k^{(0)}(t) = (1-a)(fk^{(0)}(t) + r) - w^{(0)}(t).$$

So, we got a solution to the problem P_0.

Note that the resulting function in (4.6.38), (4.6.39) coincide with the functions in formulas (4.6.26) - (4.4.28) (see. fig.4.4.2).

We arrive at the solution of the tasks P_μ. We define control $\overline{w}_\mu^*(t)$ in shape:

$$\overline{w}_\mu^*(t)=\begin{cases}\overline{w}^{(0)}(t), & 0\le t\le T,\\ \eta\left(\dfrac{T-t}{\mu}\right), & 0\le\dfrac{T-t}{\mu}\le\dfrac{T}{\mu}<\infty,\end{cases}\qquad(4.4.40)$$

where $\eta\left(\dfrac{T-t}{\mu}\right)$ – the function of boundary layer type, which have is exponential decrease. The function $\overline{w}_0(t)$ already identified, remains to be determined $\eta\left(\dfrac{T-t}{\mu}\right)$. With $\overline{w}_\mu^*(t)=\eta\left(\dfrac{T-t}{\mu}\right)$ from the second equation (4.4.29) we have:

$$\upsilon_\mu^*(t)=e^{-\frac{t}{\mu}}\left(\upsilon_0^*+\upsilon^{(0)}(0)\right)-\upsilon^{(0)}(t)-\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\eta\left(\frac{T-s}{\mu}\right)ds\,.\qquad(4.4.41)$$

With $t=T$, from (4.4.41) we have the following relation of moment:

$$c_2^*(t)=\frac{1}{\mu}\int_0^T e^{-\frac{T-s}{\mu}}\eta\left(\frac{T-s}{\mu}\right)ds,\qquad(4.4.42)$$

where $c_2^*=-\upsilon_T^*+e^{-\frac{T}{\mu}}\left(\upsilon_0^*+\upsilon^{(0)}(0)\right)-\upsilon^{(0)}(T)$.

Repeating the above procedure in this case, we define control $\overline{w}_\mu^*(t)=\eta\left(\dfrac{T-t}{\mu}\right)$ so that it is minimize the functional (4.4.11) and satisfy to relation of moment equation (4.6.42). As we saw above, such control exists on the boundary of the set (4.4.13) and getting in shape:

$$\eta\left(\frac{T-t}{\mu}\right)=\beta_1^*e^{-\frac{T-t}{\mu}}+\mu\beta_0^*e^{-\delta t}.\qquad(4.4.43)$$

Optimal coefficients β_1^*, β_0^* selected from solutions of the system algebraic equations:

$$\begin{aligned} &r_{22}^*\beta_1 + \mu r_{23}^*\beta_0 = c_2^*, \\ &r_{22}^*\mu\beta_1^{\ 2} + 2\mu^2 r_{23}^*\beta_1\beta_0 + r_{33}^*\mu^2\beta_0^{\ 2} = l, \end{aligned} \tag{4.4.44}$$

where $r_{22}^* = \lim\limits_{m_2\to -1} r_{22} = \dfrac{1}{2}\left(1 - e^{-\frac{2T}{\mu}}\right)$, $\quad r_{23}^* = \lim\limits_{m_2\to -1} r_{23} = \dfrac{1}{1-\delta\mu}\left(e^{-\delta T} - e^{-\frac{T}{\mu}}\right)$, $\quad r_{33}^* = r_{33}$.

From the first equation (4.6.44), we obtain a quadratic equation with respect to β_0^*:

$$p\beta_0^{*2} - r = 0, \tag{4.4.45}$$

where $p = r_{33}\mu^2 - \mu^3\dfrac{r_{23}^2}{r_{22}}$, $\quad r = l - \mu\dfrac{c_2^{*2}}{r_{22}}$. Equation (4.6.45) has two real roots $\beta_0^{*(1)}$ and $\beta_0^{*(2)}$. Assume that these roots exist. Then from (4.6.43) we have two functions-control, one of which minimizes the functional (4.4.11).

Substituting control (4.4.43) to (4.6.41) we obtain:

$$\upsilon_\mu^* = e^{-\frac{t}{\mu}}\left(\upsilon_0^* + \upsilon^{(0)}(0)\right) - \upsilon^{(0)}(t) - \frac{1}{2}\beta_1^* e^{-\frac{T+t}{\mu}}\left(1 - e^{\frac{2t}{\mu}}\right) - \frac{\beta_0^*\mu}{1-\delta\mu}\left(e^{-\delta t} - e^{-\frac{t}{\mu}}\right).$$

Given the (4.6.38), (4.6.43) control $\overline{w}_\mu^*(t)$ represented in the form:

$$\overline{w}_\mu^*(t) = \begin{cases} \gamma_1^{(0)} e^{m_0(T-t)} + \gamma_0^{(0)} e^{-\delta t}, & 0 \le t \le T, \\ \beta_1^* e^{-\tau} + \mu\beta_0^* e^{-\delta\mu(\tau_1 - \tau)}, & 0 \le \tau \le \tau_1 < +\infty, \end{cases}$$

where $\tau = \dfrac{T-t}{\mu}$, $\quad \tau_1 = \dfrac{T}{\mu}$.

Three $(\overline{w}_\mu^*, k^{(0)}, \upsilon_\mu^*)$ is a solution the problem. It should be noted that the system (4.4.29) approximates the system (4.4.2) with an accuracy $O(\mu)$. In addition, when $\mu \to 0$ the solution of problem P_μ approaches to solution of P_0 , which was obtained in (4.6.39) (4.6.40) (or (4.6.26) - (4.4.28)). Consequently, the solution of problem P_μ is an asymptotic approximation of the solution of the initial problem with the accuracy of the order of smallness $O(\mu)$. When the delay at enter capital investments disappears, i.e. at $\mu \to 0$ we have the solution of problem P_0. The solution of the problem P_0 forms an arterial road. The specific investment imposed by into action without delays, described by the function $\upsilon^0(t)$. In this case, we arrive to a certain idealization, but in fact the process of development of investments in the economy without delays occurs. However, the study duration of the process assimilation of capital investments in a certain period of time and the effect of the lag (delay) on other indicators of economic growth of interest of a scientific nature. In this sense, the convergence rate for solving the problem to the solution of the problem P_0 is of practical importance.

4.5 Estimation of Optimal Development of the Economy Based on a Single-Commodity Optimization Model of with a Small Parameter

Here, by the method small parameter is investigated the optimal control problem for single-commodity model of economy and conducted a comparative analysis with the known results which receive from [85] in other ways. It should be noted that at the beginning of this task is required forming arterial road of this model and to find corresponding control realizing the arterial road [20] at solving the problem (4.3.29).

$$J_1 = \int_0^T e^{-\delta t}(1-a)uxdt \to \max \tag{4.3.29}$$

or

$$J_1 = -\int_0^T e^{-\delta t}(1-a)uxdt \to \min\ . \tag{4.3.29}$$

Then the reduced problem can be formulated as follows: to find such process $\upsilon = (k(t),\ X(t), u(t))$, which minimizes the (4.3.29) at restrictions

$$\mu\dot{k} = -(\varepsilon + n)k + (1-a)(1-u)x, \tag{4.5}$$

$$k(0) = k_0\,,\ k(T) = k_1\,,$$

$$0 \le u \le 1,\ \ x = f(k,t) \ge 0\,,$$

$$k(t) \ge k_3(t)\,,\ 0 \le \mu \le 1,$$

where $f(k,t) = \frac{1}{L}F(k,L,t)$.

Here we propose a new approach, which is based on simple properties decreasing function on a closed interval. We denote by D the set of values k, satisfies (4.5) and call it the permissible region the process. A similar problem in the case of $\mu = 1$ considered in [85].

Highest average consumption, which should be secured by this process is estimated by the value of functional (4.3.29) with the sign reversed. In this problem, the state of the system is k - the amount of capital per worker, control - labor productivity x and the share of consumption u . The equation the process is the differential equation of growth capital intensity.

If enter a "rapid" time τ by formula $\tau = \dfrac{t}{\mu}$, where $\mu -$ small parameter, then time τ the original t it is a "slow" time. In such a case, the variables coefficients of the of the studied system in "fast" time τ will be slowly varying. Administering to a system small parameter μ - it is a certain idealization, which emphasizes the fact that the pace course of the process above (approximately, $\dfrac{1}{\mu}$ time increases) than in the normal mode.

Let the size of the final product are determined by the production function of the Cobb-Douglas [82, 85]. Then labor productivity x is determined by the function

$$x = be^{\rho t}k^{\alpha}, \ \alpha = 1-\beta, \tag{4.5.1}$$

where ρ - coefficient defining rate of growth of a technical process, α - the coefficient of elasticity of manufacture production assets; β -the coefficient of elasticity of the release of labor.

Equation (4.5) with (4.5.1) is written as:

$$\mu\dot{k} = -(\varepsilon + n)k + (1-a)(1-u)be^{\rho t}k^{\alpha}. \tag{4.5.2}$$

Consider the problem (4.3.29), (4.5.2). We introduce a new function

$$V = ke^{-\delta t}. \tag{4.5.3}$$

Then in view of (4.5.3) from (4.5.2) we have:

$$\mu\frac{dV}{dt} = (-(\varepsilon + n + \delta\mu)k + (1-a)be^{\rho t}k^{\alpha})e^{-\delta t} - e^{-\delta t}b(1-a)e^{\rho t}k^{\alpha}u. \tag{4.5.4}$$

Now, from the right side of (4.5.4) delete k. We require that the sum of standing to the $e^{-\delta t}$, i.e. function

$$m(k,t) = -(\varepsilon + n + \delta\mu)k + (1-a)be^{\rho t}k^{\alpha}$$

did not depend of k . Then

$$\frac{\partial m}{\partial k} = -(\varepsilon + n + \delta\mu) + (1-a)be^{\rho t}\alpha k^{\alpha-1} = 0 .$$

From here we have:

$$k^{\alpha-1} = \frac{\varepsilon + n + \delta\mu}{\alpha(1-a)be^{\rho t}} = k^{-\beta} \tag{4.5.5}$$

or

$$k = k_{\mu}^{*} = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu}\right)^{\frac{1}{\beta}} e^{\frac{\rho}{\beta}t} . \tag{4.5.6}$$

Using by (4.5.3) the equation (4.5.4) can be written in the form

$$\mu\frac{dV}{dt} = (-(\varepsilon + n + \delta\mu) + (1-a)be^{\rho t}k^{\alpha-1})V - b(1-a)e^{\rho t}uk^{\alpha-1}V . \tag{4.5.7}$$

Than in view of (4.5.5) from (4.5.7) we will obtain:

$$\mu\frac{dV}{dt} = -\frac{\varepsilon + n + \delta\mu}{\alpha}(u-\beta)V . \tag{4.5.8}$$

In view of (4.5.5) the functional (4.3.29) can be written as:

$$J = -\frac{\varepsilon + n + \delta\mu}{\alpha}\int_{0}^{T} uVdt . \tag{4.5.9}$$

Taking into account (4.5.6) from (4.5.3) we have:

$$V = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu}\right)^{\frac{1}{\beta}} e^{-(\delta-\frac{\rho}{\beta})t} = e^{-(\delta-\frac{\rho}{\beta})t}V(0), \tag{4.5.10}$$

$$V(0) = k(0) = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu} \right)^{\frac{1}{\beta}}.$$

From formulas (4.5.10) we find:

$$\dot{V} = -(\delta - \frac{\rho}{\beta})V \,. \tag{4.5.11}$$

Comparing equation (4.5.8), (4.5.11), we obtain:

$$\frac{\varepsilon + n + \delta\mu}{\alpha}(u - \beta) = \mu(\delta - \frac{\rho}{\beta}) \,. \tag{4.5.12}$$

Condition (4.5.12) takes place if

$$u = u_{\mu}^{*} = 1 - \alpha \frac{\frac{\rho}{\beta}\mu + \varepsilon + n}{\varepsilon + n + \delta\mu} \,. \tag{4.5.13}$$

Similarly in [85], the function $k_{\mu}^{*}(t)$ (4.5.6) call *highway* of the dynamic model. Control, implementing this feedline - a constant, which is determined by (4.5.13). Then in view of (4.5.13) equation (4.5.9) is written as:

$$J = -\frac{\varepsilon + n + \delta\mu}{\alpha}\left(1 - \alpha \frac{\frac{\rho}{\beta}\mu + \varepsilon + n}{\varepsilon + n + \delta\mu} \right) \int_{0}^{T} V(t)dt \,.$$

Here the integrand $V(t) = e^{-\delta t} k_{\mu}^{*}(t) = e^{-\left(\delta - \frac{\rho}{\beta}\right)t} \cdot V(0)$ - the discounted value of the capital.

From (4.5.10) can to see that at $\delta > \dfrac{\rho}{\beta}$ the function $V(t)$ - decreasing on the segment $[0, T]$, and $V(0)$ is its largest value, i.e.

$$T \cdot e^{-\left(\delta - \frac{\rho}{\beta}\right)T} V(0) \le \int_0^T V(t)dt \le T \cdot V(0).$$

Where, $J_\mu^* = J_{\min} = -\dfrac{\varepsilon + n + \delta\mu}{\alpha}\left(1 - \alpha\dfrac{\dfrac{\rho}{\beta}\mu + \varepsilon + n}{\varepsilon + n + \delta\mu}\right) \cdot V(0) \cdot T.$

With $\delta < \dfrac{\rho}{\beta}$ function $V(t)-$ increasing on the segment $[0,T]$. Then

$$T \cdot V(0) \le \int_0^T V(t)dt \le T \cdot e^{\left(\frac{\rho}{\beta} - \delta\right)T} \cdot V(0),$$

$$J_\mu^* = J_{\min} = -\frac{\varepsilon + n + \delta\mu}{\alpha}\left(1 - \alpha\frac{\dfrac{\rho}{\beta} + \varepsilon + n}{\varepsilon + n + \delta\mu}\right) \cdot T \cdot e^{\left(\frac{\rho}{\beta} - \delta\right) \cdot T} \cdot V(0). \qquad (4.5.14)$$

With $\delta = \dfrac{\rho}{\beta}$ we have

$$J_\mu^* = J_{\min} = -\frac{\varepsilon + n + \delta\mu}{\alpha}\beta \cdot T \cdot V(0). \qquad (4.5.15)$$

In the "fast" time τ highway is

$$k_\mu^* = k_\mu^*(\tau\mu) = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu}\right)^{\frac{1}{\beta}} e^{\frac{\rho}{\beta}\tau\mu} \qquad (4.5.16)$$

And it will be slowly varying functions. With $\mu \to 0$, k_μ^*, u_μ^*, J_μ^* have the following limit values

$$k_\mu^* \to k_0^* = \left(\frac{\alpha(1-a)b}{\varepsilon + n} \right)^{\frac{1}{\beta}},$$

$$u_\mu^* \to u_0^* = \beta, \quad J_\mu^* \to J_0^* = -\frac{\varepsilon + n}{\alpha} \beta V(0) \cdot T.$$

For short periods of time changing the "slow" variables does not affect the fast equations and consequently limit values are k_0^*, u_0^* may serve as an asymptotic approximation when forming highway and enables to obtain a qualitative picture of it.

In order to process υ was optimal in the sense of the solution of the task k_μ^* it should satisfy the given boundary conditions (4.5). But it not so the solution k_μ^* can not satisfy the boundary conditions (4.5) because through these points are other curves, which are partial solutions of the original equation (4.5.2) with a predetermined control u.

We define these curves and their points of intersection with the highway (switching point) k_μ^*. Dividing both sides of the differential equation (4.5) on k^α we have:

$$\mu k^{-\alpha} \dot{k} = -(\varepsilon + n) k^{1-\alpha} + (1-a)(1-u) b e^{\rho t}.$$

We introduce a new variable

$$k^{1-\alpha} = k^\beta = z. \tag{4.5.17}$$

Then, taking into account (4.5.16) from (4.5.15) we get:

$$\mu \dot{z} = -\beta(\varepsilon + n) z + \beta(1-a)(1-u) b e^{\rho t}. \tag{4.5.18}$$

With the known u (u -constant) exact solution (4.5.18) with the initial condition $z(0)=k_0^\beta$ written in the form of the Cauchy formula:

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t}{\mu}}k_0^\beta+\frac{\beta}{\mu}(1-a)(1-u)b\int_0^t e^{-\beta(\varepsilon+n)\frac{t-s}{\mu}}e^{\rho s}ds$$

or

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t}{\mu}}\left(k_0^\beta-\frac{a_0(1-u)}{\varepsilon+n+\mu\frac{\rho}{\beta}}\right)+\frac{a_0(1-u)}{\varepsilon+n+\mu\frac{\rho}{\beta}}e^{\rho t}, \qquad (4.5.19)$$

where $a_0=b(1-a)$.

Similarly, the solution of (4.5.18) with the initial condition $z(T)=k_1^\beta$ is determined by the relationship:

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t-T}{\mu}}\left(k_1^\beta-\frac{a_0(1-u)e^{\rho T}}{\varepsilon+n+\mu\frac{\rho}{\beta}}\right)+\frac{a_0(1-u)e^{\rho t}}{\varepsilon+n+\mu\frac{\rho}{\beta}}.$$

Note that if for the given problem build the Hamiltonian function, then it will depend on the control u linearly and its maximum value are achieved only in the boundary values. But in the real economic problems, as noted in [85], the minimum level of consumption is strictly positive: $0<u_*\le u\le 1$. Therefore, the Hamiltonian takes the maximum value in points $u=u_*,\quad u=1$ and through these values can be determined switching point.

For to determine the point of intersection of the highway with the boundaries of the permissible region D we have the following relations:

$$\frac{a_0\alpha}{\lambda+\delta\mu}e^{\rho t}=e^{-\beta\lambda\frac{t}{\mu}}\left(k_0^{\beta}-\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho t}, \tag{4.5.20}$$

$$\frac{a_0\alpha}{\lambda+\delta\mu}e^{\rho t}=e^{-\beta\lambda\frac{t-T}{\mu}}\left(k_1^{\beta}-\frac{a_0(1-u_i)e^{\rho T}}{\lambda+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho t}, \tag{4.5.21}$$

where $\lambda=\varepsilon+n,\qquad a_0=b(1-a)$, $i=1,2$.

In formulas (4.5.20), (5.6.21) if $i=1,$ it takes the lower limit $u=u_1=u_*$, if $i=2$, then $u=u_2=1$. Then the left and right switching points are calculated by the following formulas:

$$t_1=-\frac{\mu}{\beta\lambda+\mu\rho}\ln\frac{\dfrac{a_0\alpha}{\lambda+\delta\mu}-\dfrac{(1-u_i)a_0}{\lambda+\mu\dfrac{\rho}{\beta}}}{k_0^{\beta}-\dfrac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}},$$

$$t_2=\frac{\beta\lambda\mathrm{T}}{\beta\lambda+\mu\rho}-\frac{\mu}{\beta\lambda+\mu\rho}\ln\frac{\dfrac{a_0\alpha}{\lambda+\delta\mu}-\dfrac{(1-u_i)a_0}{\lambda+\mu\dfrac{\rho}{\beta}}}{k_1^{\beta}-\dfrac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho\mathrm{T}}}.$$

The boundaries of the permissible region D is defined by the relations (4.5.17), (4.5.19) at values $u=u_*,\quad u=1$. Let's $k_0<k_\mu^*(0),\quad k_1>k_\mu^*(T)$. Then highway $k_\mu(t)$ (4.5.15) proceeds as shown in fig. 4.5.1. As can be seen from the figure, the optimal trajectory consists of three sections with moments of switching t_1 and t_2. Starting from the time t_1 until t_2, up growth is on highway, and outside the interval (t_1, t_2) consumption is at a lower level u^*, i.e. in those periods of time in the economy is a process of accumulation. The output trajectory to a highway at different values of the small parameter is shown in fig. 4.5.3-4.5.6.

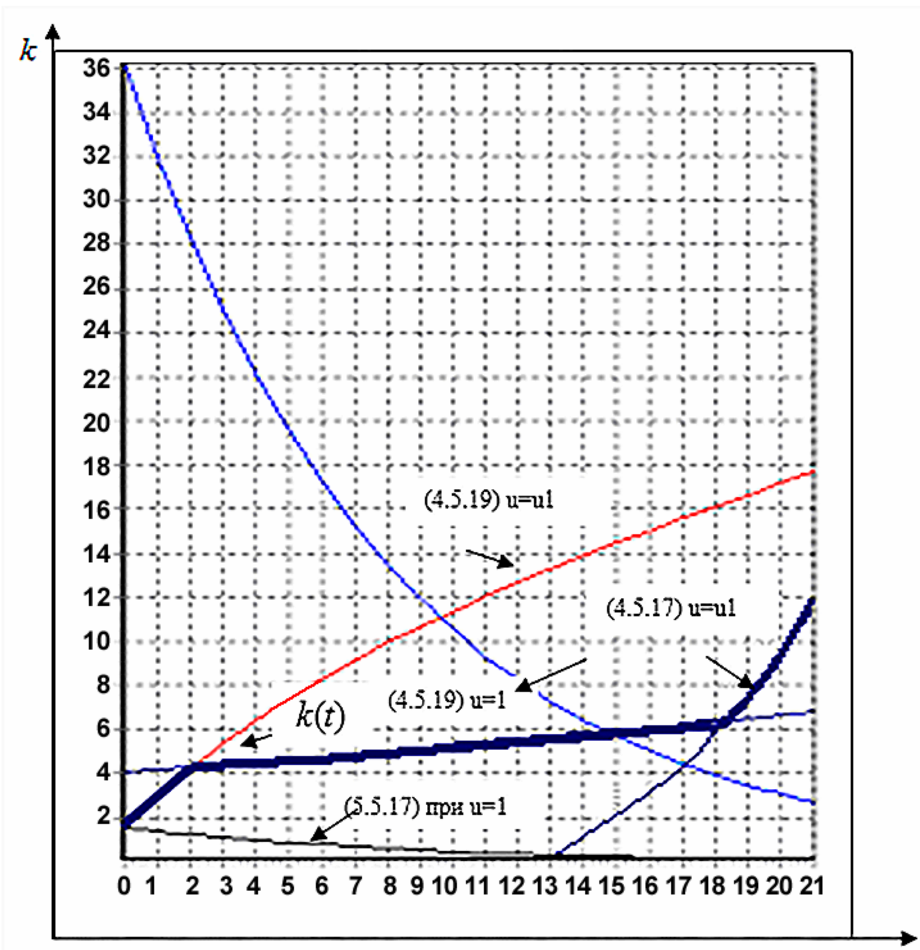

Fig. 4.5.1 *The optimal trajectory the moment of switching.*

As we noted above, that the small parameter is introduced artificially into the system, so that the result was a simplified algorithm that allows us to offer cost-effective computational procedures. Therefore, we need to derive the corresponding asymptotic formulas that make it possible to build the optimal trajectory with a certain precision, while maintaining the qualitative features of the processes under study. Moving on to "rapid" time $\tau = \dfrac{t}{\mu}$ make the change of variable in (4.5.18):

$$\frac{dz}{d\tau} = -\beta\lambda z + \beta a_0(1-u)\,,\ z(0) = k_0^{\beta}\,. \tag{4.5.22}$$

The solution of equation (4.5.22) if known u is as follows:

$$z(\tau) = e^{-\beta\lambda\tau}\left(k_0^{\beta} - \frac{a_0(1-u)}{\lambda}\right) + \frac{a_0(1-u)}{\lambda}\,. \tag{4.5.23}$$

Graph of the function is shown in fig.4.5.7. For $\sigma = \dfrac{t-T}{\mu}$ $(t > T)$ from (4.5.18) will have:

$$\frac{dz}{d\sigma} = -\beta\lambda z + \beta a_0(1-u)e^{\rho T} \quad z(0) = k_1^{\beta}\,. \tag{4.5.24}$$

The solution (4.5.24) can be written as:

$$z(\sigma) = e^{-\beta\lambda\sigma}\,(k_1^{\beta} - \frac{a_0(1-u)}{\lambda}e^{\rho T}) + \frac{a_0(1-u)e^{\rho T}}{\lambda}\,. \tag{4.5.25}$$

A graph of this function is shown in fig. 4.5.2.

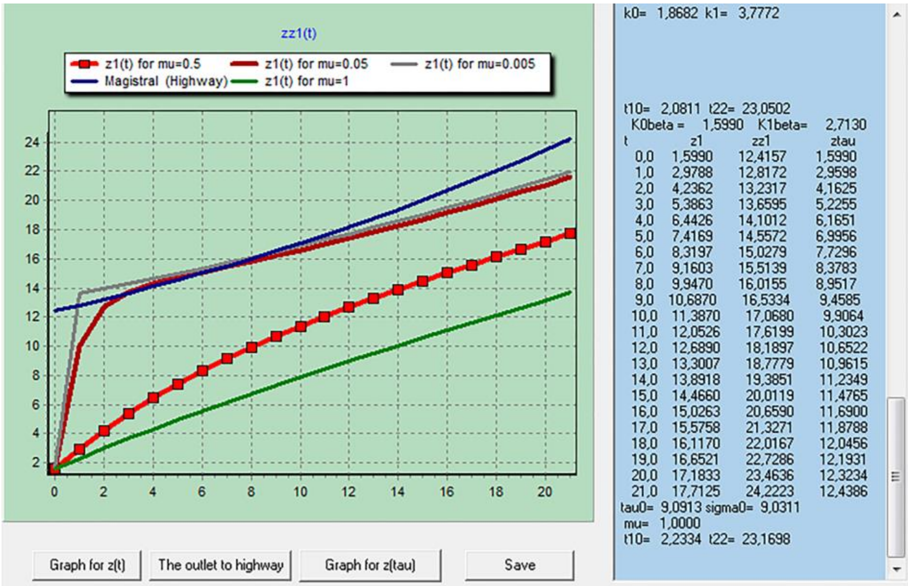

Fig. 4.5.2 *Graphics functions* $z(t)$, $z(\sigma)$ *and a table of values.*

Then we have the following asymptotic formulas defining the intersection point of highway with the boundaries of the permissible region D:

$$\tau_0 = \frac{1}{\lambda\beta}\ln\frac{\lambda k_0^{\beta} - a_0(1-u_i)}{a_0(u_i-\beta)},$$

$$\sigma_T = \frac{1}{\lambda\beta}\ln\frac{\lambda k_1^{\beta} - a_0(1-u_i)e^{\rho T}}{a_0(\alpha-(1-u_1)e^{\rho T})}.$$

At that itself highway is determined from the (4.5.15), i.e. is taken limit value k_μ at $\mu \to 0$:

$$k_0^* = \left(\frac{a_0\alpha}{\lambda}\right)^{\frac{1}{\beta}},$$

where $a_0 = b(1-a)$, $\lambda = \varepsilon + n$. It should be noted that the first term in formulas (4.5.23), (4.5.25) are, respectively, the left and right "borderline functions” [15], which approximate the transition from the initial state to the highway and go to the highway in the final state.

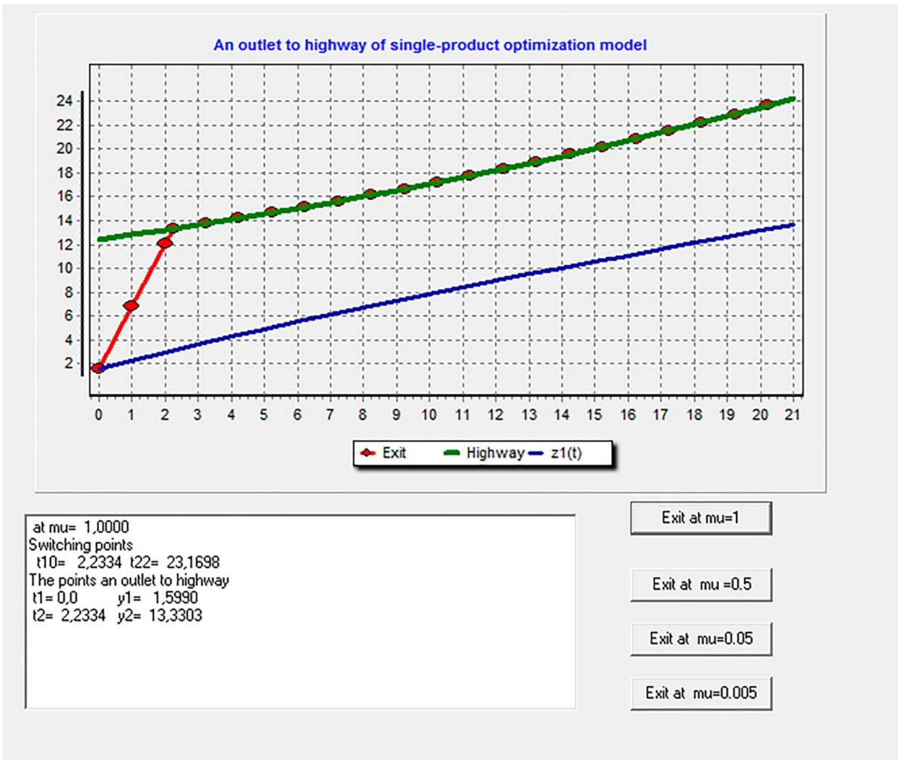

Fig. 4.5.3 *The case at* $\mu = 1$.

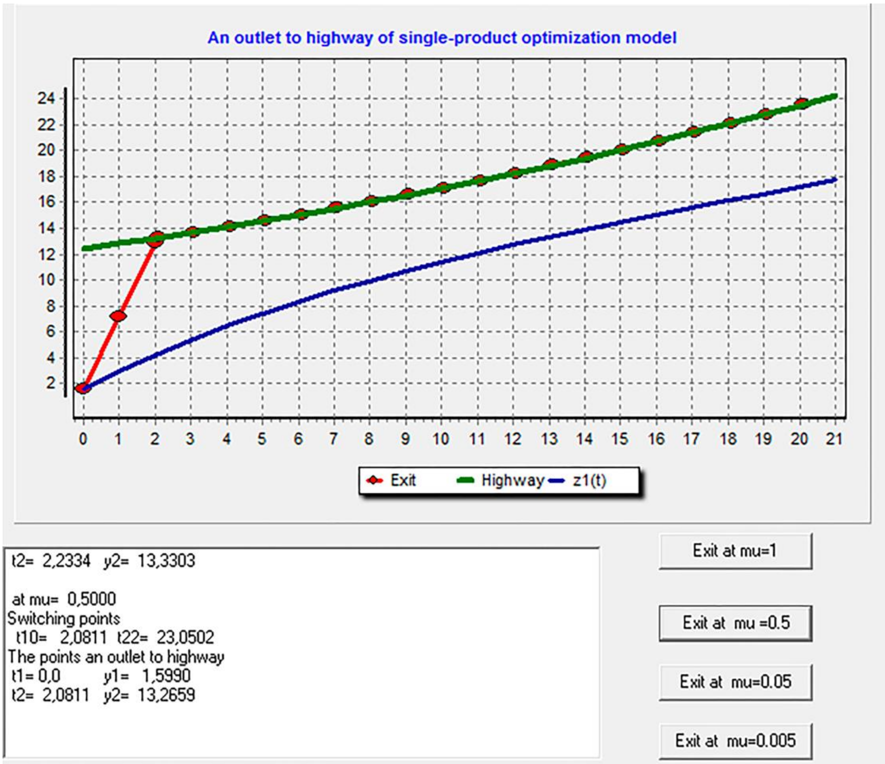

Fig. 4.5.4 *The case at* $\mu = 0.5$.

Note that if for the given problem build the function of Hamilton, then it will depend on the control u linearly and its maximum value are achieved only in the boundary values u. But in real economic problems, as noted in [94, 97], minimum level of consumption is strictly positive: $0 < u_1 \le u \le 1$. Therefore, the Hamiltonian takes the maximum value in points $u = u_1$, $u = 1$ and terms of the values can be determined switching points.

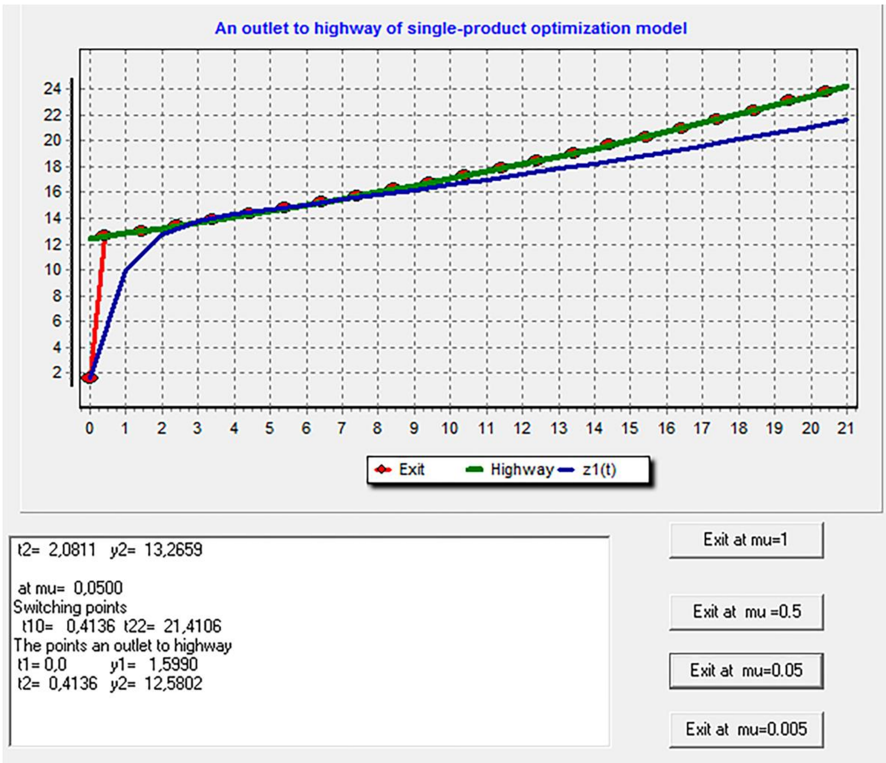

Fig. 4.5.5 *The case at* $\mu = 0.05$.

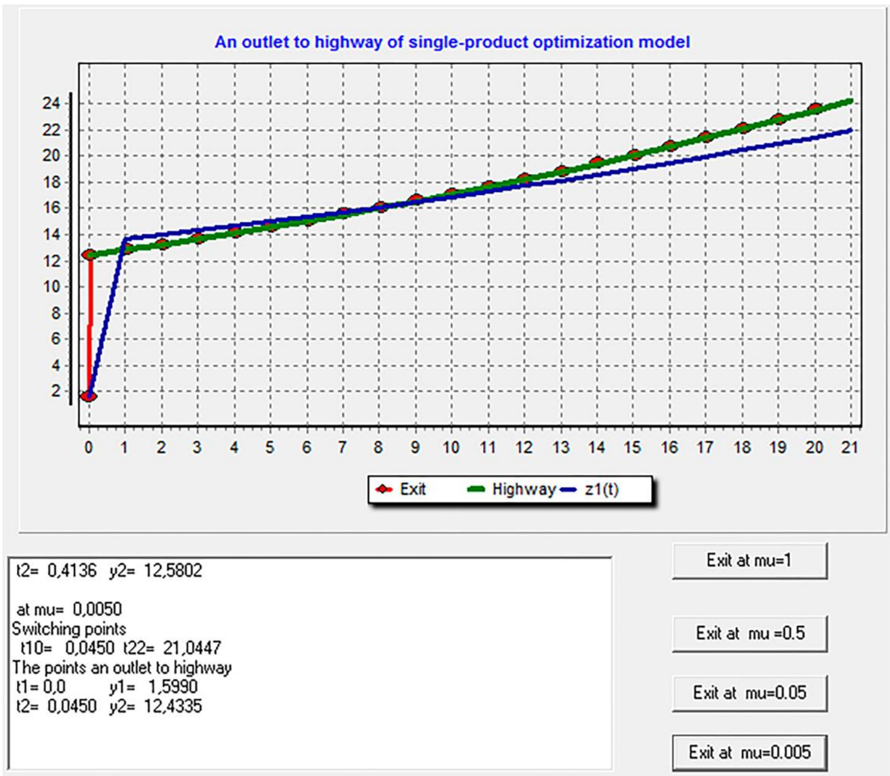

Fig. 4.5.6 *The case at* $\mu = 0.005$.

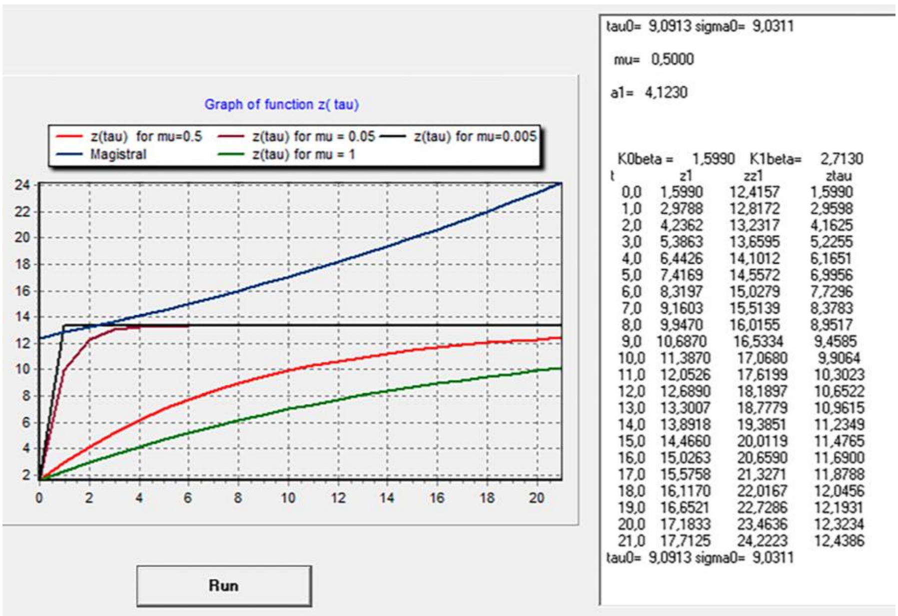

Fig. 4.5.7 *Graph of function* $z(\tau)$ *and its table of values.*