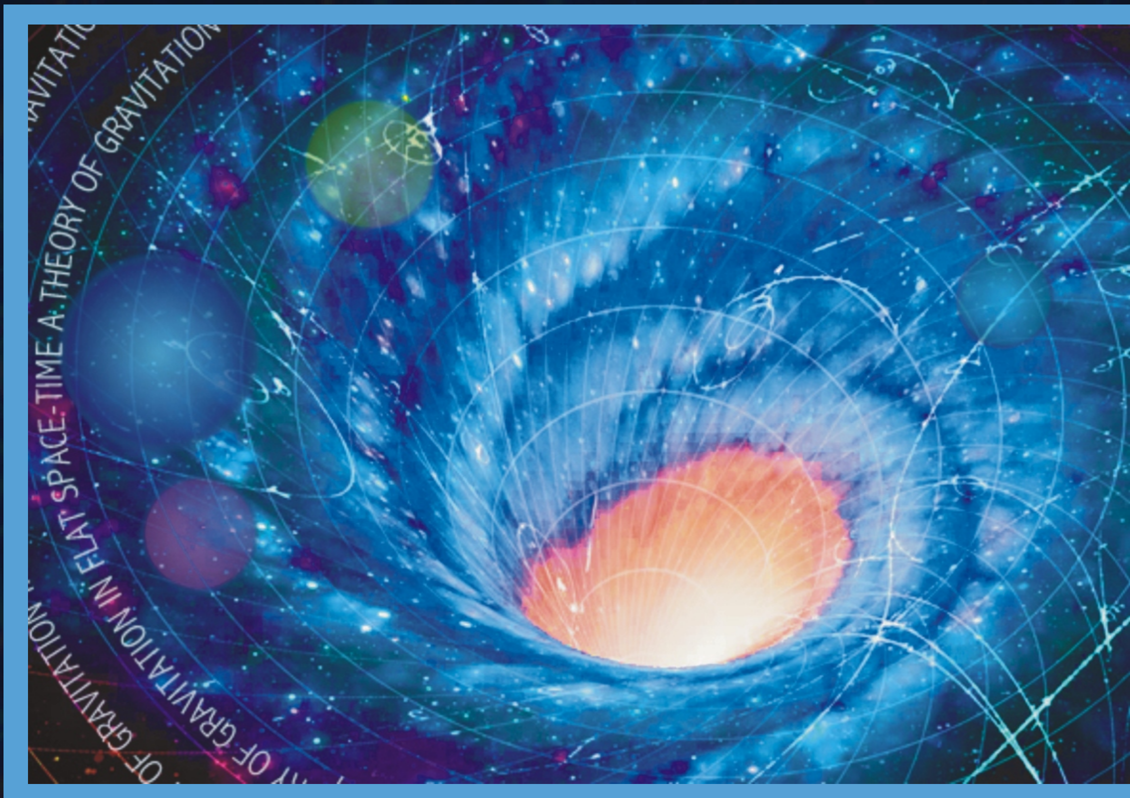

A THEORY OF GRAVITATION IN FLAT SPACE-TIME

Walter Petry

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In memory to
Albert Einstein

Dedicated to

My grandchildren

Kira Lucien
Evelyn Maya

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Introduction

Einstein's general theory of relativity is the most accepted theory of gravitation. Gravitation is explained by non-Euclidean geometry and not as field theory as e.g. the theory of Electrodynamics. The great acceptance of general relativity is based on the good predictions of many gravitational effects. The first results given already by Einstein himself are redshift, deflection of light and the perihelion in a static spherically symmetric gravitational field. Later on till now, an extensive study with different applications of general relativity had taken place. Non-stationary solutions of the theory are given, too. In particular, there are the well-known black holes and the expanding universe. In both cases singularities exist; black holes have a singularity in the centre of the body and the universe starts with a singularity in the beginning which is called "big bang". All the standard theories such as e.g. Electrodynamics are field theories whereas Einstein's theory is a geometrical theory. A book about classical field theories is e.g. given by Soper [*Sop* 76].

Therefore, I start the study of a theory of gravitation. The metric is flat space-time, e.g., the pseudo-Euclidean geometry and the gravitational potentials g_{ij} must satisfy covariant (relative to the metric) differential equations of order two. On the left hand side we have the non-linear differential operator in divergence form of the potentials whereas the total energy-momentum tensor inclusive that of gravitation is the right hand side of the differential equations. It is worth to mention that the energy-momentum of gravitation is a covariant tensor. In addition to the flat space-time metric the proper time τ is defined in analogy to the metric by a quadratic form with the potentials g_{ij} as coefficients. Such theories are already well known and are studied by many authors. They are called bi-metric theories. The first one who has studied such a theory of gravitation was Rosen [Ros 40]. Later on there were given very different bi-metric gravitational theories. Gupta [Gup 54] has the theory of Einstein written in form of a field theory by a successive approximation procedure. Kohler [Koh 52, 53] started from a flat space-time metric with several suitable Lagrangians for the gravitational field similar to our consideration. One of these Lagrangians is identical with our Lagrangian. But Papapetrou et al. [Pap 54] have given an argument against the theory of Kohler showing by linearization of the differential equations that two mass parameters appear in the potentials of the spherically symmetric gravitational field. Kohler [Koh 52, 53] constructed all the Lagrangians which yield a symmetrical energy-momentum tensor. Compare also the later article of Rosen [Ros 73] and the extensive study of Logunov and co-workers (see e. g., [Log 86]) about bi-metric theories of gravitation.

In this work the theory of gravitation in flat space-time is summarized. It was studied by the author during several years. We do not give other bi-metric theories of gravitation. Many applications of the theory of gravitation in flat space-time are studied and will be given here or at least cited where they can be found. Most of the received results of the theory of gravitation in flat space-time are compared with those of general relativity. We only give a small part of the experimental results. This work is divided into twelve chapters.

The first chapter contains the theory of gravitation in flat space-time. The energy-momentum tensor of the gravitational field is given. The field equations are in covariant form where the left hand side is a differential operator in divergence form of the gravitational field and the right hand side is the whole energy-momentum of matter and gravitation. The conservation of the whole energy-momentum is given. This law together with the field equations implies the equations of motion of matter. The field equations are also rewritten by the use of the field strength of gravitation instead of the gravitational potentials. The angular momentum of a particle and the equations of motion of the spin angular-momentum in the gravitational field are stated. Furthermore, the transformations of the equations of motion and of the spin into a uniformly moving frame are given which is used to study a gyroscope in the gravitational field of a rotating body, e.g. the Earth. This result agrees to the lowest order with the corresponding one of Einstein's theory although the used methods are quite different since gravitation in flat space-time is not a geometrical theory.

In chapter II static, spherically symmetric bodies are studied. The field equations, the equations of motion in this field and the energy-momentum are given. Inertial and gravitational mass are equal. The gravitational field in the exterior of the body is stated. This result agrees with that of Einstein's theory to some accuracy but higher order approximations deviate from one another. The case of non-singular solutions is stated and the equations of motion of a test particle in this field are given. The redshift, the deflection of light and the perihelion shift in a spherically symmetric field are received. They agree to some order with those of general relativity. Furthermore, the radar time delay is given which also agrees to the lowest order with Einstein's theory. Neutron stars are numerically studied.

In chapter III non-stationary, spherically symmetric solutions are stated. The field equations, the equations of motion and the energy-momentum conservation are given without detailed derivations. The differential equations describing the spherically symmetric body are very complicated and cannot be solved analytically; they must be solved numerically. This would be of great interest in the study of black holes.

In chapter IV rotating stars are considered. All the received results are based on numerical computations which are received by some co-workers.

In chapter V post-Newtonian approximations are calculated. The conservation law of the total energy-momentum and the equations of motion are studied. The received results again agree to the lowest order with those of general relativity.

Chapter VI contains the post-Newtonian approximations of spherically symmetry. The 1-post-Newtonian approximation agrees with the one of Einstein's theory but the 2-post-Newtonian approximations do not agree. Flat space-time theory of gravitation doesn't imply the theorem of Birkhoff. The exterior gravitational field of a non-stationary star contains small time-dependent expressions. Furthermore, the motion of a test body in the gravitational field of a non-stationary star is given. The gravitational radiation from binary stars is also studied and it is in agreement with the one of Einstein's theory.

In chapter VII homogeneous, isotropic, cosmological models are studied with and without cosmological constant. The essential result is the existence of non-singular cosmological models, i.e. there exist no "big bang" in contrast to Einstein's theory. Detailed studies of these models are given where analytic solutions can be received under the assumption that there is no cosmic microwave background radiation. In the beginning of the universe no matter exists and all the energy is in form of gravitation. In the course of time matter arises at coasts of gravitational energy. The whole energy is conserved. The universe starts with contraction to a positive value and then it expands for all times. But the two models $\Lambda > 0$ and $\Lambda = 0$ differ from one another. In the first case matter will be slowly destroyed in the course of time whereas in the second case matter in the universe increases for all times to a finite value. In this case the universe is at present time nearly stationary.

In chapter VIII the two possibilities of an expanding and a non-expanding universe are studied. The first interpretation is well known whereas the second interpretation is also possible. The interpretation of the redshift in a non-expanding universe follows by the different kinds of energy, e.g., of matter and of gravitation which are transformed into one another in the universe in the course of time. Therefore, the larger redshift of distant objects is explained by a stronger gravitational field in analogy to the redshift in a static spherically symmetric gravitational field. In addition to the standard proper time, the absolute time is introduced. The age of the universe measured with absolute time is in agreement with experimentally known results even for a vanishing cosmological constant $\Lambda = 0$.

In chapter IX post-Newtonian approximations in the universe are studied where linear, spherically symmetric perturbations are considered. In the beginning of the universe small matter density contrasts arise in the uniform distribution of matter. In the matter dominated universe the density contrast increases very fast in agreement with the observed CMBR anisotropy. General relativity gives only a small increase of the density contrast and has difficulties to explain the observed large scale structures.

In chapter X post-Newtonian approximations in the universe are studied. The gravitational potentials are computed. The equations of motion are given. The gravitational force of long-field force is compared with Newton's force. The radius of compensation of the two forces is computed, i.e., that of Newton's force and that of the long-field force are compared with one another. This radius of compensation of the two forces decreases in a universe with cosmological constant $\Lambda > 0$ and increases in a universe with cosmological constant $\Lambda = 0$.

In chapter XI preferred and non-preferred reference frames are studied. In the preferred frame the pseudo-Euclidean geometry holds and there exists an extensive study of preferred frames in the literature. In the non-preferred reference frame the velocity of light is anisotropic but for the Michelson-Morley experiment and for many other experiments the theory gives the correct results. Transformations from the different frames into one another are studied.

Chapter XII contains some additional results which are not necessary connected with the theory of gravitation in flat space-time. There are essays to explain some experimental results which are received in the last years. The first one is the anomalous flyby where the rotation of the Earth is used to study this effect. It is shown that there is a frequency jump which is not equivalent to a jump of the velocity. In this chapter the equations of Maxwell in a medium are considered, too where in addition to the pseudo-Euclidean geometry the proper-time is introduced in analogy to the theory of gravitation in flat space-time. The well-known equations of Maxwell in a medium are received. This result is subsequently generalized to the equations of Maxwell in a medium contained in the universe. We give an approximate formula for the proper-time of a medium contained in the universe. The arriving frequency of light emitted by an atom from a distant object, as e.g. galaxy or quasar contained in a medium is computed. It is applied to cosmological models. A redshift formula is received under some assumptions. In the special case that the object is not contained in a medium the well-known Hubble law holds. But more generally it may be that the assumption of dark energy is not necessary by the introduction of a medium in which photons are emitted. Galaxies or quasars with nearly the same distances can have quite different redshifts in dependence of different media in which they are contained. Furthermore, the approximate proper-time of a spherically symmetric body is stated where the universe is neglected. The velocity of a test body circulating this body is received. A simple small reflection index which depends on the distance from the centre of the body is studied. It is assumed that the medium has a fixed radius r_0 where the refraction index is equal to one for distances greater than this distance, i.e. there is no medium.

This result is applied to the Pioneers although an anisotropic emission of on-board heat may explain the observed anomalous acceleration. We get also an anomalous acceleration of the Pioneers towards the Sun with a decrease of the acceleration with increasing distance from the Sun. The application of the received velocity of a test body circulating a galaxy can also explain the observed flat rotation curves under some assumptions. The surrounding medium of the galaxy given by the refraction index may be interpreted as dark matter with radius r_0 . In this case, it follows that the mass of this dark matter surrounding the Sun is very small compared to that of the Sun whereas the mass of dark matter surrounding a galaxy is much greater than the mass of the luminous galaxy.

Summarizing it follows that for small gravitational effects the results of flat space-time theory of gravitation and of Einstein's general relativity theory agree to the measured accuracy with one another. But for the case of strong gravitational effects the two theories give quite different results. Here, we will mention the non-singular solutions of cosmological models in flat space-time theory of gravitation which means that the universe does not start with a "big bang" and the theorem of Bikhoff which does not hold in the gravitational theory of flat space-time.

No results are received about collapsing stars. Are they ending in a "black hole" or something else? The describing differential equations of a collapsing star in chapter III are very complicated and may be only solved numerically.

The violation of Bikhoff's theorem gives the hope that the star will not end in a "black hole", that is with a singularity in the centre of the star.

Contents

Chapter 1

Theory of Gravitation

In this chapter a theory of gravitation in flat space-time is studied which was considered in several articles by the author.

Let us assume a flat space-time metric. Denote by $\left(x^i\right)$ the co-ordinates of space and time then the line-element can be written

$$\left(ds\right)^2 = -\eta_{ij} dx^i dx^j \tag{1.1}$$

Here, $\left(\eta_{ij}\right)$ is a symmetric metric tensor. In addition to the metric tensor a symmetric contra-variant tensor $\left(\eta^{ij}\right)$ is defined by

$$\eta_{ik}\eta^{kj} = \delta_i^{\ j}, \quad \eta^{ik}\eta_{kj} = \delta^i_{\ j}. \tag{1.2}$$

Furthermore, we put

$$\eta = \det\left(\eta_{ij}\right). \tag{1.3}$$

In the special case of a pseudo-Euclidean metric we have

$$\left(x^i\right) = \left(x^1, x^2, x^3, ct\right). \tag{1.4}$$

$\left(x^1, x^2, x^3\right)$ are the Cartesian co-ordinates, t is the time and c is the velocity of light. Then, the metric tensor has the form

$$\left(\eta_{ij}\right) = diag\left(1,1,1,-1\right). \tag{1.5}$$

This is the metric in which the most kinds of fields and matter are described.

1.1 Gravitational Potentials

Similar to Maxwell's theory of Electrodynamics we assume that gravitation is described by a field in space and time. The electro-magnetic field can be described with the aid of a four-vector called the potentials of the field and produced by an electric four-current.

Analogously, the symmetric gravitational potentials $\left(g_{ij}\right)$ are produced by the total energy-momentum of matter and gravitational field. Similar to the equations (1.2) let us define a symmetric tensor $\left(g^{ij}\right)$ by

$$g_{ik}g^{kj} = \delta_i^{\ j}, \quad g^{ik}g_{kj} = \delta^i_{\ j} \tag{1.6}$$

We put

$$G = \det\left(g_{ij}\right). \tag{1.7}$$

In addition to the time t we define the proper-time τ by

$$c^2\left(d\tau\right)^2 = -g_{ij}dx^i dx^j. \tag{1.8}$$

The relation (1.8) is similar to the definition of the line-element (1.1) with the metric tensor $\left(\eta_{ij}\right)$. Therefore, theories of gravitation described by $\left(g_{ij}\right)$ with the proper-time (1.8) and with the line-element (1.1) are called bi-metric theories of gravitation.

1.2 Lagrangian

The theory of gravitation is derived from an invariant Lagrangian which is quadratic in the first order co-variant derivatives of the potentials $\left(g_{ij}\right)$ resp. of the contra-variant tensors $\left(g^{ij}\right)$. The derivatives are relative to the flat space-time metric (1.1) and they are denoted with a bar "/". The Lagrangian has the form

$$L_G = -\left(\frac{-G}{-\eta}\right)^{1/2} g_{kl}g_{mn}g^{ij}\left(g^{km}{}_{/i}g^{\ln}{}_{/j} - \frac{1}{2}g^{kl}{}_{/i}g^{mn}{}_{/j}\right) \tag{1.9}$$

In addition let us introduce the invariant Lagrangian

$$L_\Lambda = -8\Lambda\left(\frac{-G}{-\eta}\right)^{1/2} \tag{1.10}$$

Here, Λ is the cosmological constant. For simplicity we consider dust (no pressure) with the density ρ. The Lagrangian for matter can be written in the form

$$L_M = -\rho g_{ij} u^i u^j \tag{1.11}$$

where (u^i) is the four-velocity. It follows by the use of

$$u^i = \frac{dx^i}{d\tau} \tag{1.12}$$

and relation (1.8)

$$-g_{ij} u^i u^j = c^2 . \tag{1.13}$$

By the introducing of the constant

$$\kappa = \frac{4\pi k}{c^4} \tag{1.14}$$

the whole Lagrangian has the form:

$$L = L_G + L_\Lambda - 8\kappa L_M . \tag{1.15}$$

Here, the constant k denotes the gravitational constant.

1.3 Field Equations

The differential equations for the gravitational potentials $\left(g_{ij}\right)$ follow from the variation - equation

$$\partial \int L(-\eta)^{1/2} d^4 x . \tag{1.16}$$

From Euler's equations we get by the formulas for the covariant derivatives (see e.g., [Sop 76], p.189 ff)

$$\left[\frac{1}{(-\eta)^{1/2}} \frac{\partial L(-\eta)^{1/2}}{\partial g^{ij}{}_{/k}} \right]_{/k} = \frac{1}{(-\eta)^{1/2}} \frac{\partial L(-\eta)^{1/2}}{\partial g^{ij}} \tag{1.17}$$

implying by the use of (1.15)

$$\left[\frac{\partial L_G}{\partial g^{ij}{}_{/k}}\right]_{/k}=\frac{\partial\left(L_G+L_\Lambda\right)}{\partial g^{ij}}+8\kappa\frac{\partial L_M}{\partial g^{ij}}\,. \tag{1.18}$$

We use the following formulas

$$\frac{\partial(-G)^{1/2}}{\partial g^{ij}}=-\frac{1}{2}(-G)^{1/2}g_{ij}\,,\quad \frac{\partial g_{ij}}{\partial g^{kl}}=-g^{ik}g^{jl}\,.$$

Equation (1.18) implies by the use of these relations and multiplication with g^{lj} the following formula

$$\begin{aligned}&\left[\left(\frac{-G}{-\eta}\right)^{1/2}g^{mn}\left(g_{ik}g^{kj}{}_{/n}-\frac{1}{2}\delta_i^{\,j}g_{kl}g^{kl}{}_{/n}\right)\right]_{m}\\&=\frac{1}{2}\left(\frac{-G}{-\eta}\right)^{172}g_{mn}g_{kl}g^{jr}\left(g^{mk}{}_{/i}g^{nl}{}_{/r}-\frac{1}{2}g^{mn}{}_{/i}g^{kl}{}_{/r}\right)\\&+\frac{1}{4}\delta_i^{\,j}\left(L_G+L_\Lambda\right)+4\kappa\rho g_{im}u^m u^j\end{aligned} \tag{1.19}$$

These are the field equations of gravitation for dust.

1.4 Equations of Motion and the Energy-Momentum

We will now prove the equivalence of the conservation law of energy-momentum and the equations of motion. It follows from equation (1.18) by multiplication with $g^{kj}{}_{/l}$ and summation

$$\begin{aligned}&\left[g^{mn}{}_{/l}\frac{\partial L_G}{\partial g^{mn}{}_{/k}}\right]_{/k}-\frac{\partial L_G}{\partial g^{mn}{}_{/k}}g^{mn}{}_{/l/k}\\&=\frac{\partial\left(L_G+L_\Lambda\right)}{\partial g^{mn}}g^{mn}{}_{/l}-8\kappa\rho g^{mn}{}_{/l}g_{mr}g_{ns}u^r u^s\end{aligned}\,. \tag{1.20}$$

The mixed energy-momentum tensors of the gravitational field, of vacuum energy (given by the cosmological constant Λ) and of dust are given by

$$T(G)^j{}_i = \frac{1}{8\kappa}\left[\left(\frac{-G}{-\eta}\right)^{1/2} g_{mn}g_{kl}g^{jr}\left(g^{m,k}{}_{/i}g^{nl}{}_{/r} - \frac{1}{2}g^{mn}{}_{/i}g^{kl}{}_{/r}\right) + \frac{1}{2}\delta^j{}_i L_G\right] \tag{1.21a}$$

$$T(\Lambda)^j{}_i = \frac{1}{16\kappa}\delta^j{}_i L_\Lambda \tag{1.21b}$$

$$T(M)^j{}_i = \rho g_{im}u^m u^j \tag{1.21c}$$

and the corresponding symmetric tensors are defined by

$$T(G)^{ij} = g^{im}T(G)^j_m,\ T(\Lambda)^{ij} = g^{im}T(\Lambda)^j_m,\ T(M)^{ij} = g^{im}T(M)^j_m \tag{1.22}$$

Put

$$D^j{}_i = \left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn}g_{ik}g^{kj}{}_{/n}\right]_{/m}. \tag{1.23a}$$

Then, the field equations of gravitation (1.19) have the simple form

$$D^j{}_i - \frac{1}{2}\delta^j{}_i D^m{}_m = 4\kappa T^j{}_i \tag{1.23b}$$

Here,

$$T_i^j = T(G)_i^j + T(\Lambda)_i^j + T(M)_i^j. \tag{1.23c}$$

is the whole energy-momentum tensor of gravitational field, of vacuum energy and of matter.

The equations (1.23) can be rewritten

$$D^j{}_i = 4\kappa\left(T^j{}_i - \frac{1}{2}\delta^j{}_i T^m{}_m\right). \tag{1.24}$$

It is worth to mention that the equations (1.23) are generally co-variant. In particular, the energy-momentum of gravitation is a tensor in contrast to the corresponding pseudo-tensor in Einstein's general relativity.

The field equations of gravitation (1.23b) and (1.24) are formally similar to the corresponding equations of general relativity. Here, $D^j{}_i$ is a differential operator of order two in divergence form for g^{ij} whereas in general relativity there is instead of that the Ricci tensor. The source of the gravitational field in flat space-time theory of gravitation is the whole energy-momentum tensor inclusive the one of the gravitational field which is not a tensor in Einstein's theory and it does not appear as source for the field.

Relation (1.20) can be rewritten

$$\left[g^{mn}{}_{/k} \frac{\partial L_G}{\partial g^{mn}{}_{/l}} - \delta^l{}_k \left(L_G + L_\Lambda \right) \right]_{/l} = 8\kappa\rho g_{mn/k} u^m u^n ,$$

i.e., we get by the use of (1.21a) and (1.21b)

$$\left(T(G)^m{}_i + T(\Lambda)^m{}_i \right)_{/m} = -\frac{1}{2}\rho g_{mn/i} u^m u^n .$$

This relation becomes by the substitution of (1.21c) and the use of (1.23c)

$$T^m{}_{i/m} = T(M)^m{}_{i/m} - \frac{1}{2}\rho g_{mn/i} u^m u^n = T(M)^m{}_{i/m} - \frac{1}{2} g_{mn/i} T(M)^{mn} .$$

Hence, the conservation of the whole energy-momentum

$$T^m{}_{i/m} = 0 \tag{1.25a}$$

is equivalent with the equations of motion for matter

$$T(M)^m_{i/m} = \frac{1}{2} g_{mn/i} T(M)^{mn} \tag{1.26}$$

The conservation law of the whole energy-momentum (1.25a) can be rewritten

$$\left(\eta^{in} T^m{}_n \right)_{/m} = 0 . \tag{1.25b}$$

The conservation of mass is given by

$$\left(\rho u^m \right)_{/m} = 0 . \tag{1.27}$$

More general energy-momentum tensors for matter can be considered, e.g. the matter tensors of perfect fluid

$$T(M)^{ij} = (\rho + p)u^i u^j + pc^2 g^{ij} \tag{1.28}$$

where p denotes the pressure of matter. The conservation law of the whole energy-momentum and the equivalent equations of motion are also given by the equations (1.25) and (1.26).

The conservation law of the whole energy-momentum (1.25), the equations of motion (1.26), and the conservation law of mass (1.27) are given in co-variant form. The equations of motion (1.26) and the conservation of mass (1.27) can be rewritten in non-covariant form

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^k}\left((-\eta)^{1/2} T(M)^k{}_i\right) = \frac{1}{2}\frac{\partial g_{mn}}{\partial x^i} T(M)^{mn} \tag{1.29a}$$

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^k}\left((-\eta)^{1/2} \rho u^k\right) = 0\,. \tag{1.29b}$$

The equations (1.29) give for a test particle, i.e. $p = 0$

$$\frac{d}{d\tau}\left(g_{ik} u^k\right) = \frac{1}{2}\frac{\partial g_{mn}}{\partial x^i} u^m u^n\,. \tag{1.30}$$

It follows by differentiation, the use of (1.11), and some elementary calculations

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma(G)^i{}_{mn}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau} \tag{1.31}$$

where $\Gamma(G)^i_{mn}$ denote the Christoffel symbols of g_{ij}.

It is worth mentioning that the equations for the gravitational field can be generalized including electro-magnetic fields, scalar fields, etc., by addition of the corresponding Lagrangians for these fields to (1.15) which will not be considered.

1.5 Field Equations Rewritten

It is sometimes useful for the applications of the gravitational theory to consider instead of g^{ij} symmetric tensors defined by

$$f^{ij} = \left(\frac{-G}{-\eta}\right)^{1/2} g^{ij} \tag{1.32a}$$

and

$$f_{ij} = \left(\frac{-G}{-\eta}\right)^{-1/2} g_{ij} \tag{1.32b}$$

yielding

$$f_{1k} f^{kj} = \delta_i^{\ j}, \; f^{ik} f_{kj} = \delta^i{}_j \,. \tag{1.33}$$

Then, the equations for the gravitational field (1.23) can be rewritten

$$\left(f^{mn} f_{ik} f^{kj}{}_{/n}\right)_{/m} = 4\kappa T^j{}_i \tag{1.34}$$

where the energy-momentum tensor of gravitation has the form

$$T(G)^j{}_i = \frac{1}{8\kappa}\left[f_{mn} f_{kl} f^{jr} \left(f^{mk}{}_{/r} f^{nl}{}_{/i} - \frac{1}{2} f^{mn}{}_{/r} f^{kl}{}_{/i} \right) + \frac{1}{2}\delta^j{}_i L_G \right] \tag{1.35}$$

with

$$L_G = - f_{mn} f_{kl} f^{rs} \left(f^{mk}{}_{/r} f^{nl}{}_{/s} - \frac{1}{2} f^{mn}{}_{/r} f^{kl}{}_{/s} \right). \tag{1.36}$$

The energy-momentum tensor of perfect fluid is given by

$$T(M)^j{}_i = (\rho + p)\left(\frac{-F}{-\eta}\right)^{-1/2} f_{ik} u^j u^k + pc^2 \delta^j{}_i \tag{1.37}$$

where

$$F = \det\left(f_{ij}\right). \tag{1.38}$$

The relation (1.13) has the form

$$\left(\frac{-F}{-\eta}\right)^{-1/2} f_{mn}u^m u^n = -c^2 . \qquad (1.39)$$

1.6 Field Strength and Field Equations

The equations of motion (1.31) of a test particle in the gravitational field are not generally co-variant.

A co-variant derivative of the four-vector $\left(u^i\right)$ of a test particle is

$$\frac{Du^i}{D\tau} = \frac{du^i}{d\tau} + \Gamma^i{}_{mn}u^m u^n . \qquad (1.40)$$

$\Gamma^i{}_{mn}$ are the Christoffel symbols of the metric (1.1).

The equations of motion (1.31) can be rewritten by the substitution (1.40)

$$\frac{Du^i}{D\tau} = -\Delta\Gamma^i{}_{mn}u^m u^n \qquad (1.41)$$

where

$$\Delta\Gamma^i{}_{mn} = \Gamma\left(G\right)^i{}_{mn} - \Gamma^i{}_{mn} . \qquad (1.42)$$

Elementary calculations imply that $\Delta\Gamma^i{}_{jk}$ is a tensor of rank three. Hence, the equations of motion (1.41) for a test particle in the gravitational field $\left(g_{ij}\right)$ are generally co-variant. Similar to the equations of motion for a test particle in the electro-magnetic field where on the right hand side stands the Lorentz-force defined by the electro-magnetic field strength the tensor $\Delta\Gamma^i{}_{jk}$ in the equations (1.41) can be interpreted as gravitational field strength and the right hand side of (1.41) is the gravitational force.

Elementary calculations give

$$\frac{\partial g_{mn}}{\partial x^i} = \Gamma(G)^r{}_{mi}\, g_{nr} + \Gamma(G)^r{}_{ni}\, g_{mr} \,.$$

Hence, it follows

$$g_{mn/i} = \Delta\Gamma^r{}_{mi} g_{nr} + \Delta\Gamma^r{}_{ni} g_{mr} \,.$$

Therefore, we get

$$-g_{mk} g^{im}{}_{/j} = g^{im} g_{mk/j} = g^{im} g_{kn} \Delta\Gamma^n{}_{mj} + \Delta\Gamma^i{}_{kj} \,. \tag{1.43}$$

With the aid of (1.43) all the co-variant derivatives of g^{ij} can be replaced by the gravitational field strength. Elementary calculations give the Lagrangian

$$L_G = -2\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn} \left(\Delta\Gamma^k{}_{lm} \Delta\Gamma^l{}_{kn} + g^{kl} g_{rs} \Delta\Gamma^r{}_{km} \Delta\Gamma^s{}_{\ln} - \Delta\Gamma^r{}_{rm} \Delta\Gamma^s{}_{sn}\right) \tag{1.44}$$

The energy-momentum tensor of the gravitational field has the form

$$T(G)_i^j = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{jn} \left(\Delta\Gamma^k{}_{\ln} \Delta\Gamma^l{}_{ki} + g^{kl} g_{rs} \Delta\Gamma^r{}_{kn} \Delta\Gamma^s{}_{li} - \Delta\Gamma^r{}_{rn} \Delta\Gamma^s{}_{si}\right) + \frac{1}{16\kappa} \delta^j{}_i L_G \tag{1.45}$$

It follows for the equations of the gravitational field (1.23b)

$$\left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn} \left(\Delta\Gamma^j{}_{in} + g^{jk} g_{il} \Delta\Gamma^l{}_{kn} - \delta^j{}_i \Delta\Gamma^k{}_{kn}\right)\right]_{/m} = -4\kappa T^j{}_i \,. \tag{1.46a}$$

The field equations (1.24) have the form

$$\left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn} \left(\Delta\Gamma^j{}_{in} + g^{jk} g_{il} \Delta\Gamma^l{}_{kn}\right)\right]_{/m} = -4\kappa \left(T^j{}_i - \frac{1}{2} \delta^j{}_i T^m{}_m\right). \tag{1.46b}$$

Summarizing, we have written the theory of gravitation in flat space-time by the use of the field strength of gravitation similar to Maxwell's theory written with the aid of the electro-magnetic field strength.

1.7 Angular-Momentum

We will now derive the conservation law of the whole angular-momentum. Let us start from the conservation law of the whole energy-momentum (1.25b) which can be rewritten

$$\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left((-\eta)^{1/2}\tilde{T}^{im}\right)+\Gamma^i{}_{mn}\tilde{T}^{mn}=0 \tag{1.47a}$$

where we have introduced the non-symmetric energy-momentum tensor

$$\tilde{T}^{ij}=\eta^{im}T^j{}_m\,. \tag{1.47b}$$

In an inertial frame, i.e. the metric tensor (η_{ij}) is constant and therefore $\Gamma^i{}_{jk}=0$ the relation (1.47a) implies a conservation law of the whole energy-momentum. Therefore, we get

$$P^i=\int(-\eta)^{1/2}\tilde{T}^{14}d^3x \quad \text{(i=1-4)} \tag{1.48}$$

Where P^i is a constant and the integration is taken over the whole space. Equation (1.47a) gives

$$\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left(x^j(-\eta)^{1/2}\tilde{T}^{im}\right)=\tilde{T}^{ij}-x^j\Gamma^i{}_{mn}\tilde{T}^{mn}\,. \tag{1.49}$$

The field equations (1.23) imply

$$\tilde{T}^{ij}=\eta^{im}T^j{}_m=\frac{1}{4\kappa}\left[\left(\frac{-G}{-\eta}\right)^{1/2}g^{mn}\left(\eta^{ik}g_{kl}g^{jl}{}_{/n}-\frac{1}{2}\eta^{ij}g_{kl}g^{kl}{}_{/n}\right)\right]_{/m}\,.$$

The substitution of this relation into equation (1.49) and the subtraction from the arising from the same equation where i and j are exchanged yields

$$\begin{aligned}&\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left(x^j(-\eta)^{1/2}\tilde{T}^{im}-x^i(-\eta)^{1/2}\tilde{T}^{jm}\right)\\&=A^{ijm}{}_{/m}-\left(x^j\Gamma^i{}_{mn}-x^i\Gamma^j{}_{mn}\right)\tilde{T}^{mn}\end{aligned} \tag{1.50}$$

with the contra-variant tensor

$$A^{ijk} = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{km} g_{rs}\left(\eta^{is} g^{jr}{}_{/m} - \eta^{js} g^{ir}{}_{/m}\right). \tag{1.51}$$

It follows from equation (1.50) by the use of relations for the co-variant derivatives of tensors of order three

$$\begin{aligned} &\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}((-\eta)^{1/2}\left(x^i \tilde{T}^{jm} - x^j \tilde{T}^{im} + A^{ijm}\right)) \\ &= \left(x^j \Gamma^i{}_{mn} - x^i \Gamma^j{}_{mn}\right)\tilde{T}^{mn} - \Gamma^i{}_{mn} A^{njm} - \Gamma^j{}_{mn} A^{inm} \end{aligned}. \tag{1.52}$$

These equations imply in uniformly moving frames the conservation law of the angular-momentum, i.e.

$$M^{ij} = \int (-\eta)^{1/2}\left[x^i \tilde{T}^{j4} - x^j \tilde{T}^{i4} + A^{ij4}\right] d^3x \;\left(i, j = 1,2,3,4\right) \tag{1.53}$$

is constant for all times. The first two expressions correspond to the usual definition of the angular momentum. To study the last expression we use the first part of the relation (1.43) and rewrite (1.51)

$$A^{ijk} = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g_{mn} g^{kr} g^{sm}{}_{/r}\left(\eta^{in}\delta^j{}_s - \eta^{jn}\delta^i{}_s\right). \tag{1.54}$$

We now define the canonical momentum

$$\Pi_{ij} = \frac{1}{16\kappa}\frac{\partial L_G}{\partial g^{ij}{}_{/4}} \tag{1.55a}$$

implying

$$\Pi_{ij} = -\frac{1}{8\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{4k} g^{mn}{}_{/k}\left(g_{im} g_{jn} - \frac{1}{2} g_{ij} g_{mn}\right). \tag{1.55b}$$

The Hamiltonian is given by

$$H = \Pi_{mn} g^{mn}{}_{/4} - \frac{1}{16\kappa} L_G\,. \tag{1.56a}$$

Elementary computations give

$$H = -T(G)^4{}_4, \tag{1.56b}$$

i.e. H is the energy density of the gravitational field. It follows from (1.54) by the use of relation (1.55b)

$$A^{ij4} = 2\left(\delta^j{}_m \delta^i{}_n - \delta^i{}_m \delta^j{}_n\right)\eta^{mk} g^{nl} \Pi_{kl}. \tag{1.57}$$

We define for $i, j = 1, 2, 3, 4$ the anti-symmetric four-matrices

$$\Sigma^{ij} = \left(\Sigma^{ij}{}_{mn}\right) = \left(\delta^j{}_m \delta^i{}_n - \delta^i{}_m \delta^j{}_n\right) \tag{1.58}$$

with the proper-values $0, \pm i$. The relation (1.57) can be rewritten

$$A^{ij4} = 2\Sigma^{ij}{}_{mn} \eta^{mk} g^{nl} \Pi_{lk}. \tag{1.59}$$

Hence, the last expression in equation (1.53) of the angular momentum can be interpreted as consequence of the spin of the gravitational field.

1.8 Equations of the Spin Angular Momentum

In this sub-chapter we follow along the lines of Papapetrou [Pap 51] who uses a method of Fock [Foc 39]. The following detailed calculations can be found in [Pet 91].

The equations of motion for matter (1.29a) can be written in the form:

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^m}\left((-\eta)^{1/2} T(M)^{im}\right) = -\Gamma(G)^i{}_{mn} T(M)^{mn} \tag{1.60}$$

where it is assumed that $T(M)^{ij}$ vanishes outside of a narrow tube which surrounds the world line of the test particle. The test particle describes a world line $X(t) = \left(X^i(t)\right)$ with $X^4(t) = ct$. Let us put in analogy to [Pap 51]

$$M^{ij} = u^4 \int (-\eta)^{1/2} T(M)^{ij} d^3x \tag{1.61a}$$

$$M^{kij} = -u^4 \int \left(x^k - X^k(t) \right) (-\eta)^{1/2} T(M)^{ij} d^3x \tag{1.61b}$$

$$\gamma^{ij} = -\frac{1}{u^4} \left(M^{ij4} - M^{ji4} \right). \tag{1.61c}$$

We obtain the equations of motion

$$\frac{d}{d\tau} \left(\frac{M^{i4}}{u^4} \right) = -\Gamma(G)^i{}_{mn} M^{mn} + \frac{\partial}{\partial x^k} \left(\Gamma(G)^i{}_{mn} M^{kmn} \right) \tag{1.62a}$$

and of the spin angular momentum

$$\begin{aligned} &\frac{d}{d\tau} \gamma^{ij} + \frac{u^i}{u^4} \frac{d}{d\tau} \gamma^{j4} - \frac{u^j}{u^4} \frac{d}{d\tau} \gamma^{i4} \\ &= \left(\Gamma(G)^i{}_{mn} - \frac{u^i}{u^4} \Gamma(G)^4{}_{mn} \right) M^{jmn} \\ &+ \left(\Gamma(G)^j{}_{mn} - \frac{u^j}{u^4} \Gamma(G)^4{}_{mn} \right) M^{imn} \end{aligned} \tag{1.62b}$$

Furthermore, we have

$$2M^{ijk} = -\left(\gamma^{ij} u^k + \gamma^{ik} u^j \right) + \frac{u^i}{u^4} \left(\gamma^{4j} u^k + \gamma^{4k} u^j \right) \tag{1.63a}$$

$$M^{ij4} + M^{ji4} = -u^i \gamma^{j4} - u^j \gamma^{i4} \tag{1.63b}$$

$$M^{i44} = -u^4 \gamma^{i4} \tag{1.63c}$$

$$\begin{aligned} M^{ij} &= \frac{u^i}{u^4} \left(\frac{u^j}{u^4} M^{44} - \frac{d}{d\tau} \left(\frac{M^{j44}}{u^4} \right) - \Gamma(G)^4{}_{mn} M^{jmn} \right) \\ &- \frac{d}{d\tau} \left(\frac{M^{ij4}}{u^4} \right) - \Gamma(G)^j{}_{mn} M^{imn}. \end{aligned} \tag{1.63d}$$

Some of the relations (1.62) are identities. Therefore, we have eight equations (four equations (1.62a), three equations (1.62b) and one equation (1.13) for the eleven unknowns quantities M^{44}, u^i $(i=1,2,3,4)$, and γ^{ij} $(i,j=1,2,3)$. It is proved in [Pap 51] that γ^{ij} is the components of a tensor and the expression

$$m=-\frac{1}{c^3}\frac{1}{u^4}\left(M^{m4}+\Gamma(G)^m{}_{kl}\,\gamma^{k4}u^l\right)u_m \tag{1.64}$$

is a scalar where $u_i=g_{im}u^m$. We will now give a co-variant formulation of the equations (1.62).

In analogy to (1.40) we define the co-variant derivative

$$\frac{D}{D\tau}\gamma^{ij}=\gamma^{ij}{}_{/m}u^m=\frac{d}{d\tau}\gamma^{ij}+\Gamma^i{}_{mn}\gamma^{nj}u^m+\Gamma^j{}_{mn}\gamma^{in}u^m. \tag{1.65}$$

Let us introduce the anti-symmetric tensor

$$A^{ij}=\frac{D}{D\tau}\gamma^{ij}+\Delta\Gamma^i{}_{mn}\gamma^{mj}u^n+\Delta\Gamma^j{}_{mn}\gamma^{im}u^n. \tag{1.66}$$

Then, we have by (1.62b), (1.63a), (1.42) and (1.65)

$$A^{ij}+\frac{u^i}{u^4}A^{j4}-\frac{u^j}{u^4}A^{i4}=0. \tag{1.67}$$

When we multiply (1.67) with u_j we get

$$\frac{1}{u^4}A^{i4}=-\frac{u^m}{c^2}\left(\frac{u^i}{u^4}A^{m4}+A^{im}\right). \tag{1.68}$$

By the use of the last two relations we get the co-variant form of (1.62b)

$$A^{ij}+\frac{1}{c^2}u_m\left(u^jA^{im}-u^iA^{jm}\right)=0. \tag{1.69}$$

We will now give (1.62a) in co-variant form and write (1.63d) for $j=4$ with the aid of (1.63a), (1.63c), $M^{4ij}=0$, (1.65) and (1.66)

$$M^{i4}+\Gamma(G)^i{}_{mn}\,\gamma^{m4}u^n=A^{i4}+\frac{u^i}{u^4}\left(M^{44}+\Gamma(G)^4{}_{mn}\,\gamma^{m4}u^n\right). \tag{1.70}$$

We get by multiplying this relation with $\frac{u^i}{u^4}$ and the use of (1.64)

$$\frac{1}{\left(u^4\right)^2}\left(M^{44}+\Gamma\left(G\right)^4{}_{mn}\gamma^{m4}u^n\right)=mc+\frac{1}{c^2}\frac{u_m}{u^4}A^{m4}. \tag{1.71}$$

Hence, we get from (1.70) by the use of (1.71) and (1.68)

$$\frac{1}{u^4}M^{i4}=mcu^i-\frac{1}{u^4}\Gamma\left(G\right)^i{}_{kl}\gamma^{k4}u^l-\frac{1}{c^2}u_kA^{ik}.$$

Now, it follows from (1.62a) by the use of (1.68), (1.61) and elementary calculations

$$\begin{aligned}&\frac{d}{d\tau}\left(mcu^i-\frac{1}{c^2}u_kA^{ik}\right)+\Gamma\left(G\right)^i{}_{kl}u^k\left(mcu^l-\frac{1}{c^2}u_rA^{lr}\right)\\&+\left(\frac{\partial}{\partial x^m}\Gamma\left(G\right)^i{}_{lk}+\Gamma\left(G\right)^i{}_{nm}\Gamma\left(G\right)^n{}_{kl}\right)\gamma^{mk}u^l\\&=0\end{aligned} \tag{1.72}$$

The introduction of the co-variant derivative of a four-vector gives

$$\begin{aligned}&\frac{D}{D\tau}\left(mcu^i-\frac{1}{c^2}u_mA^{im}\right)+\Delta\Gamma^i{}_{mn}u^m\left(mcu^n-\frac{1}{c^2}u_kA^{nk}\right)\\&+\frac{1}{2}R^i{}_{mnk}\gamma^{nm}u^k=0\end{aligned} \tag{1.73}$$

where $R^i{}_{mnk}$ is the curvature tensor of g_{ij}. Although the equations (1.62a) and (1.62b) are identical with those of general relativity the co-variant forms (1.73), (1.69) together with (1.66) are different from those of general relativity [Pap 51]. γ^{ij} which is defined by (1.61c) is not the spin in flat space-time theory of gravitation. The spin of a particle must be defined by

$$\begin{aligned}S^{ij}=&\int\left(x^i-X^i\left(t\right)\right)\left(-\eta\right)^{1/2}T\left(\tilde{M}\right)^{j4}d^3x\\&-\int\left(x^j-X^j\left(t\right)\right)\left(-\eta\right)^{1/2}T\left(\tilde{M}\right)^{i4}d^3x\end{aligned}. \tag{1.74}$$

In Einstein's theory the motion of a spin in free fall can be described according to the equations of parallel transport (see e.g. [Wei 72]. This is not possible by the use of flat space-time theory of gravitation.

1.9 Transformation to Co-Moving Frame

In the previous sub-chapter we have seen that there are not enough equations for the spin components. Schiff [Sch 80] remarked that one has to transform the equations of spin components to the co-moving frame, i.e. to the frame of the gyroscope. We use the considerations of Petry [Pet 86] to transform from a preferred frame Σ' with $(\eta_{ij}') = diag(1,1,1,-1)$ to a non-preferred frame Σ moving with velocity $v' = (v^1{}', v^2{}', v^3{}')$ relative to the frame Σ'. Let $\left(X^1{}'(t'), X^2{}'(t'), X^3{}'(t')\right)$ be the distance vector of Σ from Σ'. Then,

$$\frac{d}{dt'} X^i{}'(t') = -v^i{}' . \tag{1.75}$$

The transformations of quantities in Σ' to the corresponding ones in the co-moving frame Σ are given in [Pet 86]

$$x^i{}' = x^i + \left(\gamma^{-1} - 1\right) \frac{\left(x, \frac{v'}{c}\right)}{\left|\frac{v'}{c}\right|^2} \frac{v^i{}'}{c} + X^i{}'(t'), \quad dt' = \gamma dt \tag{1.76a}$$

with

$$\gamma = \left(1 - \left|\frac{v'}{c}\right|^2\right)^{-1/2} . \tag{1.76b}$$

It is sufficient to consider (1.76) up to quadratic expressions in the absolute value of the velocity $|v'|$, i.e.

$$x^i{}' \approx x^i - \frac{1}{2}\left(x, \frac{v'}{c}\right)\frac{v^i}{c} + X^i{}', \; dt' \approx \left(1 + \frac{1}{2}\left|\frac{v'}{c}\right|^2\right) dt . \tag{1.77}$$

In the frame Σ' we consider equation (1.62b), multiplied with $d\tau/dt'$, the use of (1.63a) and $u^i{}'/u^4{}'=v^i{}'/c$, i.e.

$$\begin{aligned}&\frac{d}{dt'}\gamma^{ij}{}'+\frac{v^i{}'}{c}\frac{d}{dt'}\gamma^{j4}{}'-\frac{v^j{}'}{c}\frac{d}{dt'}\gamma^{i4}{}'-\left(\Gamma(G)^i{}'_{mn}-\frac{v^i{}'}{c}\Gamma(G)^4{}'_{mn}\right)\gamma^{jm}{}'v^n{}'\\&+\left(\Gamma(G)^j{}'_{mn}-\frac{v^j{}'}{c}\Gamma(G)^4{}'_{mn}\right)\gamma^{im}{}'v^n{}'\\&+\left(\frac{v^j{}'}{c}\Gamma(G)^i{}'_{mn}-\frac{v^i{}'}{c}\Gamma(G)^j{}'_{mn}\right)\gamma^{4m}{}'v^n{}'=0\end{aligned}\tag{1.78}$$

Furthermore, it follows

$$\gamma^{ij}{}'=\gamma^{mn}\frac{\partial x^i{}'}{\partial x^m}\frac{\partial x^j{}'}{\partial x^n}.$$

We get after some calculations for the spin tensor γ^{ij} in Σ

$$\gamma^{ij}{}'\approx\gamma^{ij}+\frac{v^i{}'}{c}\gamma^{4j}+\frac{v^j{}'}{c}\gamma^{i4}-\frac{1}{2}\left(\frac{v^i{}'}{c}\sum_{k=1}^{3}\gamma^{kj}\frac{v^k{}'}{c}+\frac{v^j{}'}{c}\sum_{k=1}^{3}\gamma^{ik}\frac{v^k{}'}{c}\right)\tag{1.79a}$$

$$\gamma^{i4}{}'\approx\left(1+\frac{1}{2}\left|\frac{v'}{c}\right|^2\right)\gamma^{i4}-\frac{1}{2}\frac{v^i{}'}{c}\sum_{k=1}^{3}\gamma^{k4}\frac{v^k{}'}{c}.\tag{1.79b}$$

If we substitute (1.79) into (1.78) and neglect expressions of the form $\Gamma^{\bullet}{}_{\bullet\bullet}\frac{v^{\bullet}}{c}\frac{v^{\bullet}}{c}$ it follows by elementary calculations

$$\frac{d}{dt'}\gamma^{ij}=\Gamma(G)^4{}_{44}\left(-v^i{}'\gamma^{j4}+v^j{}'\gamma^{i4}\right)-\left(\sum_{k=1}^{3}\Omega^{ik}\gamma^{jk}-\Omega^{jk}\gamma^{ik}\right)\tag{1.80a}$$

where

$$\Omega^{ij}=-\left(\sum_{k=1}^{3}\Gamma(G)^i{}_{jk}v^k{}'+\Gamma(G)^i{}_{j4}c-\Gamma(G)^4{}_{j4}v^i{}'+\frac{1}{2}\Gamma(G)^i{}_{44}v^j{}'+\frac{1}{2}\Gamma(G)^j{}_{44}v^i{}'\right)\tag{1.80b}$$

We will now apply the result to the spin angular momentum of a test particle in the gravitational field of a spherically symmetric body in the preferred frame

Σ' with mass M and angular velocity ω. It holds in Σ' up to linear approximations

$$\begin{aligned} g_{ij}' &= \delta_{ij}\left(1+2\frac{kM}{c^2 r}\right),(i,j=1,2,3) \\ &= -\left(1-2\frac{kM}{c^2 r}\right),(i=j=4) \\ &= -\frac{2kJ}{c^3}\frac{1}{r^3}[\omega\times x'],(i=4,j=1,2,3;i=1,2,3,j=4) \end{aligned} \tag{1.81}$$

where J is the momentum of inertia. We get by elementary computations

$$r=\left(\sum_{k=1}^{3}\left|x^k\right|^2\right)^{1/2},\Omega^{11}=\Omega^{22}=\Omega^{33}\approx\frac{kM}{c^2}\frac{1}{r^3}(x',v'),\Omega^{ij}=-\Omega^{ji},i\neq j.$$

Put

$$\Omega=\left(\Omega^{23},\Omega^{31},\Omega^{12}\right)$$

then, we have

$$\Omega\approx-\frac{3}{2}\frac{kM}{c^2}\frac{1}{r^3}[x'\times v']+\frac{kJ}{c^2}\frac{1}{r^3}\left(\omega-3\frac{(x',\omega)}{r^2}x'\right). \tag{1.82}$$

We define $\gamma=\left(\gamma^{23},\gamma^{31},\gamma^{12}\right)$. Relation (1.81a) is rewritten in the form

$$\frac{d}{dt'}\gamma\approx2\frac{kM}{c^2r^3}(x',v')\gamma-[\Omega\times\gamma]. \tag{1.83a}$$

By the use of the law of Newton

$$\frac{dv'}{dt'}\approx-kM\frac{x'}{r^3}$$

we get

$$\frac{d}{dt'}\gamma\approx-2\left(\frac{1}{c}\frac{dv'}{dt'},\frac{v'}{c}\right)\gamma-[\Omega\times\gamma]$$

$$\Omega \approx \frac{3}{2}\left[\frac{1}{c}\frac{dv'}{dt'}\times\frac{v'}{c}\right]+\frac{kJ}{c^2r^3}\left(\omega-3\frac{(x',\omega)}{r^2}x'\right). \tag{1.83b}$$

We consider instead of γ the spin. We get in Σ by the use of the standard transformation formula, considering only expressions which are quadratic in the velocity and linear in the expression $\frac{kM}{c^2r}$, the use of (1.81)

$$\begin{aligned} g_{ij} &\approx \delta_{ij}\left(1+2\frac{kM}{c^2r}\right)-\frac{v^{i\,\prime}}{c}\frac{v^{j\,\prime}}{c},\ (i,j=1,2,3) \\ &\approx \frac{v^{i\,\prime}}{c},\ (i=1,2,3;\ j=4) \\ &\approx \frac{v^{j\,\prime}}{c},\ (i=4;\ j=1,2,3) \\ &\approx -\left(1-2\frac{kM}{c^2r}\right)(i=j=4). \end{aligned} \tag{1.84}$$

The metric tensor has the form

$$\begin{aligned} \eta_{ij} &= \delta_{ij},\ \ (i,j=1,2,3,4) \\ &= \frac{v^{i\,\prime}}{c},\ \ (i=1,2,3;\ j=4) \\ &= \frac{v^{j\,\prime}}{c},\ \ (i=4;\ j=1,2,3) \\ &= -\left(1-\left|\frac{v'}{c}\right|^2\right),\ \ (i=j=4). \end{aligned} \tag{1.85}$$

We get from the definition (1.74), (1.85) and (1.61) for i,j=1,2,3

$$\begin{aligned} S^{ij} &= \eta^{jm}g_{mk}\int x^i(-\eta)^{1/2}T(M)^{k4}\,d^3x-\eta^{im}g_{mk}\int x^j(-\eta)^{1/2}T(M)^{k4}\,d^3x \\ &\approx \left(1+2\frac{kM}{c^2r}\right)\gamma^{ij}. \end{aligned}$$

Hence, it holds

$$\gamma \approx \left(1 - 2\frac{kM}{c^2 r}\right) S . \tag{1.86}$$

We have by the substitution of (1.86) into the relation (1.83a)

$$\frac{d}{dt'}\left(\left(1 - 2\frac{kM}{c^2 r} + \left|\frac{v'}{c}\right|^2\right) S\right) \approx \left(1 - 2\frac{kM}{c^2 r}\right)[\Omega \times S] .$$

By the use of the conservation law of energy

$$\frac{1}{2}\left|\frac{v'}{c}\right|^2 - \frac{kM}{c^2 r} = const$$

we get

$$\frac{d}{dt} S \approx -[\Omega \times S] . \tag{1.87}$$

Equation (1.87) gives the precession of the spin of a test particle with constant angular velocity. It agrees with the corresponding result of general relativity [Sch 60]. The angular momentum of a gyroscope processes without changing in magnitude. The results about the spin angular momentum and the gyroscope agree with those of general relativity.

All these results of the sub-chapters 1.8 and 1.9 can be found in [Pet 91]. For experimental technical problems compare Will [Wil 81].

The results of chapter I about the theory of gravitation in flat space-time can be found in the articles of Petry [Pet 79, 81a, 82,93b].

It is worth to mention the article [Cah 07] of Cahill who has studied a theory of gravitation with application to cosmology by a method which is totally different from general relativity and any bi-metric theory.

1.10 Approximate Solution in Empty Space

By the use of general relativity approximate solutions in empty space are received by linearization of the non-linear equations. This can also be considered by the use of flat space-time theory of gravitation as will be seen in sub-chapter 2.2. Therefore, we will study the linearization of the gravitational field. We start from the gravitational theory in flat space-time (1.23) together with the conservation of the whole energy-momentum (1.25). Formula (1.23b) of the field equations implies by the use of covariant differentiation, the conservation law (1.25a) and the use of the pseudo-Euclidean geometry (1.5)

$$\frac{\partial}{\partial x^j} D_i^j - \frac{1}{2}\frac{\partial}{\partial x^i} D_m^m = 0 \quad \text{(i=1-4).} \tag{1.88}$$

Relation (1.88) gives by the use of linearization, i.e.

$$g^{ij} = \eta^{ij} + \Delta g^{ij}$$

the linearized expression

$$D_i^j = \eta^{mn}\eta_{ik}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\Delta g^{kj}.$$

Therefore, relation (1.88) can be written in the form

$$\eta^{mn}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\left\{\eta_{ik}\frac{\partial}{\partial x^j}\Delta g^{kj} - \frac{1}{2}\frac{\partial}{\partial x^i}\left(\eta_{kl}\Delta g^{kl}\right)\right\} = 0.$$

The operator in front of the bracket is the wave operator. Hence we get

$$\eta_{ik}\frac{\partial}{\partial x^j}\Delta g^{kj} - \frac{1}{2}\frac{\partial}{\partial x^i}\left(\eta_{kl}\Delta g^{kl}\right) \text{ (i=1-4).} \tag{1.89}$$

Relation (1.89) is identical with the result of general relativity (see e.g.[Rob 68], p. 256, [Sex 83], $p.$175) which is used for many applications.The derivation of relation (1.89) in empty space (no matter) uses the fact that in empty space a gravitational field exists which must be considered. The quite different study of linear approximations of the gravitational field by flat space-time theory of gravitation and general relativity follows from the different sources in the theories. Flat space-time theory of gravitation has the whole energy-momentum as source whereas general relativity has only the matter tensor. In general relativity the energy-momentum is not a tensor which implies many difficulties (see the extensive study of Logunov and co-workers (see e.g. [Log 86], [Den 82,84]).

A comparison of the theory of gravitation in flat space-time and the theory of general relativity is given in [Pet 14a].

Chapter 2

Static Spherical Symmetry

In this chapter the theory of gravitation in flat space-time stated in the previous chapter I is applied to static spherically symmetric problems with the matter tensor of a perfect fluid.

It is useful to introduce spherical polar coordinates (r,ϑ,ϕ) with

$$x^1 = r\sin\vartheta\cos\phi,\ x^2 = r\sin\vartheta\sin\phi,\ x^3 = r\cos\vartheta. \tag{2.1}$$

We get by simple computations

$$\eta_{11} = 1,\ \eta_{22} = r^2,\ \eta_{33} = r^2\sin^2\vartheta,\ \eta_{44} = -1,\ \eta_{ij} = 0\ (i \neq j). \tag{2.2}$$

Then, we have

$$(-\eta)^{1/2} = r^2\sin\vartheta.$$

The non-vanishing Christoffel symbols of the metric are

$$\begin{gathered}\Gamma^1{}_{22} = -r, \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r},\ \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r},\ \Gamma^1{}_{33} = -r\sin^2\vartheta, \\ \Gamma^2{}_{33} = -\sin\vartheta\cos\vartheta,\ \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\vartheta\end{gathered}. \tag{2.3}$$

2.1 Field Equations, Equations of Motion and Energy-Momentum

The potentials are written in the form

$$\begin{gathered}g^{11} = f(r), g^{22} = \frac{g(r)}{r^2}, g^{33} = \frac{g(r)}{r^2\sin^2\vartheta}, \\ g^{44} = -h(r), g^{ij} = 0, (i \neq j)\end{gathered} \tag{2.4a}$$

It follows

$$\begin{gathered}g_{11} = \frac{1}{f}, g_{22} = \frac{r^2}{g}, g_{33} = \frac{r^2\sin^2\vartheta}{g}, \\ g_{44} = -\frac{1}{h}, g_{ij} = 0, (i \neq j)\end{gathered} \tag{2.4b}$$

We get

$$(-G)^{1/2} = \frac{r^2 \sin\vartheta}{g(fh)^{1/2}}, \quad \left(\frac{-G}{-\eta}\right)^{1/2} = \frac{1}{g(fh)^{1/2}}, \tag{2.5}$$

For a body at rest we have $u^1 = u^2 = u^3 = 0$, i.e. it follows from relation (1.13)

$$u^4 = ch^{1/2}. \tag{2.6}$$

Then, the matter tensor of a perfect fluid (1.28) is given by

$$\begin{aligned} T(M)^i{}_j &= pc^2, (i = j = 1,2,3) \\ &= -\rho c^2, (i = j = 4) \\ &= 0. (i \neq j) \end{aligned} \tag{2.7}$$

We get from the equations (1.21a) and (1.9) by the use of (2.4) and (2.5) the energy-momentum tensor of the gravitational field

$$\begin{aligned} T(G)^i{}_j &= -\frac{1}{16\kappa}(L_1 - L_2), (i = j = 1) \\ &= \frac{1}{16\kappa} L_1, \ (i = j = 2,3) \\ &= \frac{1}{16\kappa}(L_1 + L_2), \ (i = j = 4) \\ &= 0, \ (i \neq j) \end{aligned} \tag{2.8}$$

Here,

$$L_1 = -\frac{f}{g(fh)^{1/2}}\left(\left(\frac{f'}{f}\right)^2 + 2\left(\frac{g'}{g}\right)^2 + \left(\frac{h'}{h}\right)^2 - \frac{1}{2}\left(\frac{f'}{f} + 2\frac{g'}{g} + \frac{h'}{h}\right)^2\right), \tag{2.9a}$$

$$L_2 = -\frac{4}{r^2}\frac{f}{g(fh)^{1/2}}\left(\frac{f-g}{f}\right)^2, \tag{2.9b}$$

$$L = L_1 + L_2, \tag{2.9c}$$

where the prime $'$ denotes differentiation with regard to the distance r. The field equations (1.24) with $\Lambda = 0$ give by the use of the covariant derivatives the following three equations:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{f}{g(fh)^{1/2}}\frac{f'}{f}\right)-\frac{2}{r^2}\frac{f}{g(fh)^{1/2}}\frac{f^2-g^2}{f^2}=-\frac{1}{2}L_1+2\kappa c^2(\rho-p), \tag{2.10a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{f}{g(fh)^{1/2}}\frac{g'}{g}\right)+\frac{1}{r^2}\frac{f}{g(fh)^{1/2}}\frac{f^2-g^2}{f^2}=-\frac{1}{4}L_2+2\kappa c^2(\rho-p), \tag{2.10b}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{f}{g(fh)^{1/2}}\frac{h'}{h}\right)=-2\kappa c^2(\rho+3p). \tag{2.10c}$$

The conservation law of the energy-momentum (1.25a) implies

$$\frac{d}{dr}(L_2-L_1)-\frac{4}{r}L_2+16\kappa c^2\frac{d}{dr}p=0.$$

It follows by multiplication with r^3

$$\frac{d}{dr}\left[r^3(L_2-L_1)\right]=r^2(L_1+L_2)-16\kappa c^2 r^3\frac{d}{dr}p. \tag{2.11}$$

The equations of motion (1.29a) yield

$$\frac{d}{dr}p=-\frac{1}{2}\left(\frac{f'}{f}+2\frac{g'}{g}\right)p+\frac{1}{2}\frac{h'}{h}\rho. \tag{2.12}$$

In addition to the equations (2.10), (2.11) and (2.12) we have an equation of state

$$p=p(\rho). \tag{2.13}$$

The natural boundary conditions are for $r\rightarrow\infty$

$$f(r)\rightarrow 1,\ g(r)\rightarrow 1,\ h(r)\rightarrow 1 \tag{2.14a}$$

and for $r\rightarrow 0$

$$r^2\frac{f}{g(fh)^{1/2}}\frac{f'}{f}\rightarrow 0,\ r^2\frac{f}{g(fh)^{1/2}}\frac{g'}{g}\rightarrow 0,\ r^2\frac{f}{g(fh)^{1/2}}\frac{h'}{h}\rightarrow 0. \tag{2.14b}$$

2.2 Gravitational and Inertial Mass

Let us assume a spherically symmetric star with radius r_0 . Then, the boundary condition of the pressure has the form

$$p\left(r_0\right)=0 . \qquad (2.15)$$

The mass and the pressure are defined by

$$M=4\pi\int_0^{r_0} r^2\rho\left(r\right)dr\,,\ P=4\pi\int_0^{r_0} r^2 p\left(r\right)dr\ . \qquad (2.16)$$

We get from (2.10c) with the aid of the boundary conditions (2.14) for $r>r_0$

$$r^2\frac{f}{g\left(fh\right)}\frac{h'}{h}=-2\frac{k\left(M+3P\right)}{c^2} \qquad (2.17)$$

where (1.14) is used. Equation (2.17) gives by integration and the boundary conditions (2.14) for $r>r_0$

$$h\left(r\right)=1+2\frac{k\left(M+3P\right)}{c^2}\frac{1}{r}+O\left(\frac{1}{r^2}\right). \qquad (2.18)$$

Equation (2.18) implies the gravitational mass

$$M_g=M+3P\ . \qquad (2.19)$$

The inertial mass M_i is given by

$$M_ic^2=-4\pi\int\left(T\left(M\right)^4{}_4+T\left(G\right)^4{}_4\right)r^2dr \qquad (2.20)$$

It follows by the use of (2.7), (2.8) and (2.16)

$$M_i=M-\frac{c^2}{16k}\int_0^{\infty} r^2\left(L_1+L_2\right)dr\ . \qquad (2.21)$$

We put by virtue of (2.14a) and $r>>r_0$

$$f = 1 - 2\frac{\alpha}{r} + O\left(\frac{1}{r^2}\right), g = 1 - 2\frac{\beta}{r} + O\left(\frac{1}{r^2}\right). \tag{2.22}$$

Equation (2.10) gives by integration and the use of (2.14b)

$$r^2 \frac{f}{g(fh)^{1/2}} \frac{g'}{g} + 2\int_0^r \frac{f}{g(fh)^{1/2}} \frac{(f-g)g}{f^2} dr = 2\kappa c^2 \int_0^r r^2 (\rho - p) dr .$$

It follows for $r \to \infty$ with the aid of (2.18), (2.22) and (2.16)

$$\int_0^\infty \frac{f}{g(fh)^{1/2}} \frac{(f-g)g}{f^2} dr = \frac{k}{c^2}(M - P) - \beta . \tag{2.23}$$

The existence of the integral of equation (2.23) gives by using (2.18) and (2.22) $\alpha = \beta$,i.e., we have

$$f = 1 - 2\frac{\alpha}{r} + O\left(\frac{1}{r^2}\right),\ g = 1 - 2\frac{\alpha}{r} + O\left(\frac{1}{r^2}\right). \tag{2.24}$$

We assume the natural boundary conditions as $r \to \infty$

$$r^3 L_1 \to 0,\quad r^3 L_2 \to 0,\quad r^3 p \to 0 .$$

Then, equation (2.11) implies by integration

$$\begin{aligned} r^3 (L_2 - L_1) = \int_0^r r^2 (L_1 + L_2) dr - 16\kappa c^2 r^3 p(r) \\ +48\kappa c^2 \int_0^r r^2 p(r) dr. \end{aligned} \tag{2.25}$$

Hence, we get for $r \to \infty$ by the use of (2.18), (2.24), (2.15) and (2.16)

$$\int_0^\infty r^2 (L_1 + L_2) dr = -48 \frac{k}{c^2} P . \tag{2.26}$$

Substituting equation (2.26) into (2.21) it follows with equation (2.19)

$$M_i = M + 3P = M_g , \tag{2.27}$$

i.e., inertial and gravitational mass are identical.

In general relativity the definition of inertial mass gives difficulties by virtue of the non-covariance of the energy-momentum of the gravitational field (see e.g. [Dem 82]).

In particular, equation (2.26) can be rewritten

$$-\frac{4\pi}{c^2}\int_0^\infty r^2 T(G)^4{}_4 dr = 3P. \tag{2.28}$$

Equation (2.12) together with (2.10c) implies that there exists no spherically symmetric star without pressure.

We get by a suitable linear combination of the equations (2.10) and by integration using the boundary conditions (2.14b)

$$\begin{aligned} & r^2 \frac{f}{g(fh)^{1/2}}\left(\frac{f'}{f}+2\frac{g'}{g}+3\frac{h'}{h}\right) \\ & = -\frac{1}{2}\int_0^r r^2(L_1+L_2)dr - 24kc^2\int_0^r r^2 p(r)dr. \end{aligned} \tag{2.29}$$

Hence, we have for $r \to \infty$ by virtue of (2.26), (2.17) and (2.24)

$$\alpha = \frac{k}{c^2}(M+3P) = \frac{k}{c^2}M_g. \tag{2.30}$$

Put

$$K = \frac{kM_g}{c^2} \tag{2.31}$$

then, we have for $r >> r_0$

$$\begin{aligned} & f = 1 - 2\frac{K}{r} + O\left(\left(\frac{K}{r}\right)^2\right), \ g = 1 - 2\frac{K}{r} + O\left(\left(\frac{K}{r}\right)^2\right), \\ & h = 1 + 2\frac{K}{r} + O\left(\left(\frac{K}{r}\right)^2\right). \end{aligned} \tag{2.32}$$

Equation (2.23) gives

$$\int_0^\infty \frac{f}{g(fh)^{1/2}} \frac{(f-g)g}{f^2} dr = -4\frac{k}{c^2}P. \tag{2.33}$$

The gravitational field in the exterior of the spherically symmetric star with pressure is given to the first order approximation by (2.32), i.e. by one mass, namely the gravitational mass M_g. This is similar to Einstein's general theory of relativity in contrast to Rosen's bi-metric gravitation theory where the field is described by two mass parameters M_g and M' with $M_g \neq M'$ for non-vanishing pressure.

2.3 Gravitational Field in the Exterior

Let us study the gravitational field in the exterior of the star, i.e. $r > r_0$. We have from (2.10a), (2.10b) and (2.17) with the definitions (2.19) and (2.31)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{f}{g(fh)^{1/2}} \frac{f'}{f}\right) - \frac{2}{r^2}\frac{f}{g(fh)^{1/2}} \frac{f^2-g^2}{f^2} = -\frac{1}{2}L_1 \tag{2.34a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{f}{g(fh)^{1/2}} \frac{g'}{g}\right) + \frac{1}{r^2}\frac{f}{g(fh)^{1/2}} \frac{f^2-g^2}{f^2} = -\frac{1}{4}L_2 \tag{2.34b}$$

$$r^2 \frac{f}{g(fh)^{1/2}} \frac{h'}{h} = -2K. \tag{2.34c}$$

Substituting

$$\xi = K/r$$

into equations (2.34) we get by elementary computations

$$\frac{d}{d\xi}\left(\frac{f_\xi}{f}\right) = \frac{2}{\xi^2}\left(1-\left(\frac{g}{f}\right)^2\right) - \frac{1}{4}\left(\frac{f_\xi}{f}\right)^2 + \frac{1}{4}\left(\frac{h_\xi}{h}\right)^2 - \frac{g_\xi}{g}\frac{h_\xi}{h} \tag{2.35a}$$

$$\frac{d}{d\xi}\left(\frac{g_\xi}{g}\right) = -\frac{2}{\xi^2}\left(1-\frac{g}{f}\right)\frac{g}{f} + \left(\frac{g_\xi}{g}\right)^2 - \frac{1}{2}\frac{f_\xi}{f}\frac{g_\xi}{g} + \frac{1}{2}\frac{g_\xi}{g}\frac{h_\xi}{h} \tag{2.35b}$$

$$\frac{h_\xi}{h} = 2g\left(\frac{h}{f}\right)^{1/2} \tag{2.35c}$$

where, the index ξ means the derivative relative to ξ. Put

$$f = \exp(x+z), g = \exp(y+z), h = \exp(-z). \tag{2.36}$$

Then, it follows from (2.35)

$$x_{\xi\xi} = \frac{2}{\xi^2}\left(1-\exp\left(2(y-x)\right)\right) + 4\exp(2y-x) - \frac{1}{4}\left(x_\xi\right)^2 \tag{2.37a}$$

$$y_{\xi\xi} = -\frac{2}{\xi^2}\left(1-\exp(y-x)\right)\exp(y-x) - \frac{1}{2}x_\xi y_\xi + \left(y_\xi\right)^2 \tag{2.37b}$$

$$z_\xi = -2\exp(y-x/2). \tag{2.37c}$$

The equations (2.35) and (2.32) imply for $\xi \to 0$

$$x = O\left(\xi^3\right), y = O\left(\xi^2\right), z = -2\xi + O\left(\xi^2\right).$$

Substituting the approximations of x and y up to the order four in ξ into the equations (2.37a) and (2.37b) we get by elementary calculations

$$x \approx 2A\xi^3 + \frac{1}{6}\xi^4, \quad y \approx \xi^2 - A\xi^3 + \frac{2}{3}\xi^4 \tag{2.38a}$$

where A is an arbitrary parameter which must be fixed by the interior solution. Equation (2.37c) together with (2.38a) yields

$$z \approx -2\xi - \frac{2}{3}\xi^3 + A\xi^4. \tag{2.38b}$$

Finally, we obtain from (2.36) and (2.38) up to order four in $\frac{K}{r}$:

$$f \approx 1-2\frac{K}{r}+2\left(\frac{K}{r}\right)^2-(2-2A)\left(\frac{K}{r}\right)^3+\left(\frac{13}{6}-3A\right)\left(\frac{K}{r}\right)^4 \tag{2.39a}$$

$$g \approx 1-2\frac{K}{r}+3\left(\frac{K}{r}\right)^2-(4+A)\left(\frac{K}{r}\right)^3+\left(\frac{31}{6}+3A\right)\left(\frac{K}{r}\right)^4 \tag{2.39b}$$

$$h \approx 1+2\frac{K}{r}+2\left(\frac{K}{r}\right)^2+2\left(\frac{K}{r}\right)^3+(2-A)\left(\frac{K}{r}\right)^4. \tag{2.39c}$$

Elementary computations give up to order five in K

$$\begin{aligned} L_1 &\approx -8\frac{K^2}{r^4}-8\frac{K^4}{r^6}+40A\frac{K^5}{r^7}, \\ L_2 &\approx -4\frac{K^4}{r^6}+24A\frac{K^5}{r^7}, \\ L_G &\approx -8\frac{K^2}{r^4}-12\frac{K^4}{r^6}+64A\frac{K^5}{r^7}. \end{aligned} \tag{2.40}$$

It is easily proved that the conservation law of energy-momentum (2.11) holds to the considered accuracy.

Einstein's theory gives in harmonic coordinates

$$f_E = \frac{1-K/r}{1+K/r} \approx 1-2\frac{K}{r}+2\left(\frac{K}{r}\right)^2-2\left(\frac{K}{r}\right)^3+2\left(\frac{K}{r}\right)^4 \tag{2.41a}$$

$$g_E = \frac{1}{(1+K/r)^2} \approx 1-2\frac{K}{r}+3\left(\frac{K}{r}\right)^2-4\left(\frac{K}{r}\right)^3+5\left(\frac{K}{r}\right)^4 \tag{2.41b}$$

$$h_E = \frac{1+K/r}{1-K/r} \approx 1+2\frac{K}{r}+2\left(\frac{K}{r}\right)^2+2\left(\frac{K}{r}\right)^3+2\left(\frac{K}{r}\right)^4. \tag{2.41c}$$

The solution in the exterior of the star by Einstein's theory does not contain a free parameter. The results of the two theories agree for f and g up to the order two and for h up to the order three in the case $A \neq 0$ and for $A=0$ the agreement of the solutions for f and g is up to the order three and for h up to the order four. Hence, we have high agreement of the exterior solutions of both theories.

We will now give a lower limit for the pressure of stars on the assumption that K/r_0 is small. Let us assume a non-negative density of the gravitational energy in the interior of the body, i.e.

$$-T(G)^4{}_4 \geq 0 \text{ for } r \leq r_0$$

then, it follows by the use of (2.26), (2.8) and (2.40)

$$3P = \frac{4\pi}{c^2}\int_0^\infty r^2\left(-T(G)^4{}_4\right)dr \geq \frac{4\pi}{c^2}\int_{r_0}^\infty r^2\left(-T(G)^4{}_4\right)dr$$
$$= \frac{c^2}{16k}\left(8\frac{K^2}{r_0} + 4\frac{K^4}{r_0{}^3} - 16A\frac{K^5}{r_0{}^4}\right).$$

Hence, we have by the use of (2.31)

$$\frac{K}{r_0} + \frac{1}{2}\left(\frac{K}{r_0}\right)^3 - 2A\left(\frac{K}{r_0}\right)^4 \leq 6\frac{P}{M_g}. \qquad (2.42)$$

Inequality (2.42) gives for our Sun ($M_\oplus \approx 1.993 \bullet 10^{33} g$, $r_\oplus \approx 6.96 \bullet 10^{10} cm$)

$$P_\oplus / M_\oplus \geq 3.6 \bullet 10^{-7}.$$

Numerical methods are used to obtain the solution in the exterior of the star for large values of $\xi = K/r$. For small ξ $\left(\leq 10^{-2}\right)$ the solutions (2.38) and (2.39) are used. The system of the differential equations (2.37) is numerically solved by the use of Runge-Kutta methods for different values of the parameter A. There are two different types of solutions: (1) regular solutions, i.e. for all $\xi \geq 0$ the functions f, g and h exist and are positive. This is the case for all values $A \geq 0.2$. (2) Singular solutions, i.e. it exists a positive value ξ_c depending on A such that f, g and h do not exist or vanish at ξ_c. Case (2) arises for small positive and all negative values of A.

2.4 Non-Singular Solutions

We will now study the solution in the vicinity of the singularity

$$\frac{f_\xi}{f} \approx \frac{\alpha}{\xi_c - \xi}, \ \frac{g_\xi}{g} \approx \frac{\beta}{\xi_c - \xi}, \ \frac{h_\xi}{h} \approx \frac{\delta}{\xi_c - \xi} \tag{2.43}$$

with suitable constants α, β and γ. This gives near the singularity $\xi < \xi_c$

$$f \approx \frac{A_0}{(\xi_c - \xi)^\alpha}, \ g \approx \frac{B_0}{(\xi_c - \xi)^\beta}, h \approx \frac{C_0}{(\xi_c - \xi)^\delta} \tag{2.44}$$

with some constants A_0, B_0 and C_0. We get by the substitution of (2.43) and (2.44) into the equation (2.35c)

$$\frac{\delta}{\xi_c - \xi} = 2B_0 \left(\frac{C_0}{A_0}\right)^{1/2} \frac{1}{\left(\xi_c - \xi\right)^{\beta + (\delta - a)/2}}$$

implying

$$\beta + \left(\delta - \alpha\right)/2 = 1, \ \delta = 2B_0 \left(C_0 / A_0\right)^{1/2} > 0 \,. \tag{2.45a,b}$$

It follows by the substitution of (2.43) and (2.44) into (2.35b) and the use of (2.45a)

$$\beta - \alpha < 1 \,. \tag{2.45c}$$

We have from (2.35a)

$$\alpha = \frac{1}{4}\delta^2 - \frac{1}{4}\alpha^2 - \beta\delta \,. \tag{2.45d}$$

The equations (2.45a) and (2.45d) yield by elementary calculations

$$1 + 3\beta^2 - 4\beta - 2\alpha\beta = 0$$

Hence, we get

$$\beta \neq 0, \ \alpha = \frac{1 + 3\beta^2 - 4\beta}{2\beta}, \ \delta = \frac{1 - \beta^2}{2\beta} \,. \tag{2.46}$$

We obtain by (2.46) and (2.45c)

$$\beta > 0$$

implying by the use of (2.46) and (2.45)

$$0 < \beta < 1. \tag{2.47}$$

Hence, we have

$$\alpha = \frac{(1-3\beta)(1-\beta)}{2\beta}, \ \delta = \frac{1-\beta^2}{2\beta}, \ 0 < \beta < 1 \tag{2.48a}$$

$$B_0 \left(\frac{C_0}{A_0} \right)^{1/2} = \frac{\delta}{2}. \tag{2.48b}$$

Therefore, the constants β and δ are always positive whereas α is positive for $\beta < 1/3$, negative for $\beta > 1/3$ and zero for $\beta = 1/3$. The radial velocity of light v_L near the critical value ξ_c is given by

$$v_l = c\left(\frac{f}{h}\right)^{1/2} \approx c\left(\frac{A_0}{C_0}\right)^{1/2} \left(\xi_c - \xi\right)^{1-\beta} \to 0 \tag{2.49}$$

for $\xi \to \xi_c$ by the use of (2.48a).

The solutions (2.44) cannot be continued to $\xi > \xi_c$ by virtue of (2.48a). This is similar to Rosen's bi-metric theory of gravitation $[Ros74]$. Therefore, static spherically symmetrical stars with radius $r_0 < K/\xi_c = r_c$ cannot exist in this gravitational theory.

We will now study a static spherically symmetric star with the radius $r_0 = r_c$. We get from (2.43), (2.44) and (2.48) for $r \to r_c$

$$r^2 \frac{f}{g(fh)^{1/2}} \frac{f'}{f} \to -2\frac{kM_g}{c^2} \frac{1+3\beta^2-4\beta}{1-\beta^2}$$

$$r^2 \frac{f}{g(fh)^{1/2}} \frac{g'}{g} \to -2\frac{kM_g}{c^2} \frac{2\beta^2}{1-\beta^2}$$

$$r^2 \frac{f}{g(fh)^{1/2}} \frac{h'}{h} \to -2\frac{kM_g}{c^2}.$$

Therefore, we have as $r \to r_c$

$$r^2 \frac{f}{g(fh)^{1/2}} \left(\frac{f'}{f} + 2\frac{g'}{g} + 3\frac{h'}{h} \right) \to -8\frac{kM_g}{c^2} \frac{1+\beta^2-\beta}{1-\beta^2}.$$

The left hand side of (2.29) is continuous, i.e. this equation gives

$$-8\frac{kM_g}{c^2} \frac{1+\beta^2-\beta}{1-\beta^2} = 8\kappa \int_0^{r_0} r^2 \left(-\overset{G}{T}{}^4{}_4 \right) dr - 24\frac{kP}{c^2} \geq -24\frac{kP}{c^2}.$$

Hence, we get by virtue of (2.19)

$$M_g \left(1+\beta^2-\beta\right) \leq 3P\beta\left(1-2\beta\right). \tag{2.50}$$

The assumption $P=0$ implies by virtue of (2.48a) that the mass $M_g = 0$. Therefore, we have $P>0$. Relation (2.50) can be rewritten

$$(M-P)\left(1+\beta^2-\beta\right) \leq -P\left(1-4\beta+7\beta^2\right) < 0,$$

i.e. we obtain

$$M < P. \tag{2.51}$$

An equation of state with velocity of sound c_s has the form

$$p = c_s{}^2 \rho, \quad c_s{}^2 \leq 1.$$

Hence, we get by integration the inequality

$$P \leq M$$

which is in contradiction to (2.51).

Therefore, every static spherically symmetric star has a radius $r_0 > K/\xi_c$, i.e. static spherically symmetric bodies have no singular solutions.

In empty space a singularity at a Euclidean distance from the centre can exist.

The radius of this singular sphere is smaller than the radius of the body. Hence, there is no event horizon, i.e. static black holes do not exist. Escape of energy and information is possible, i.e. no contradiction to quantum mechanics (see [Pet 14b]). It is worth to mention that the singularity -if it exists- is at a Euclidean distance and is not a singularity of the coordinate system as by general relativity.

2.5 Equations of Motion

In this sub-chapter the equations of motion of a test particle in a spherically symmetric gravitational field are studied.

Let us assume that the particle is moving in the plane given by the coordinates x^1 and x^2, i.e. $\vartheta = \pi / 2$. The velocity is given in spherical polar coordinates by

$$\left(\frac{dr}{dt}, 0, \frac{d\phi}{dt} \right). \tag{2.52}$$

The equations (1.30) for a test particle can be written by the use of (2.4b)

$$\frac{d}{dt}\left(\frac{1}{f} \frac{dr}{dt} \frac{dt}{d\tau} \right) = \frac{1}{2}\left(-\frac{f'}{f^2}\left(\frac{dr}{dt} \right)^2 + \frac{r}{g}\left(2 - r\frac{g'}{g} \right)\left(\frac{d\varphi}{dt} \right)^2 + \frac{h'}{h^2}c^2 \right)\frac{dt}{d\tau} \tag{2.53a}$$

$$\frac{d}{dt}\left(\frac{r^2}{g} \frac{d\varphi}{dt} \frac{dt}{d\tau} \right) = 0 \tag{2.53b}$$

$$\frac{d}{dt}\left(\frac{1}{h} \frac{dt}{d\tau} \right) = 0 . \tag{2.53c}$$

The relation (1.13) has the form

$$c^2 \left(\frac{d\tau}{dt} \right)^2 = \frac{c^2}{h} - \frac{1}{f}\left(\frac{dr}{dt} \right)^2 - \frac{r^2}{g}\left(\frac{d\varphi}{dt} \right)^2 . \tag{2.54}$$

Equation (2.53c) yields

$$\frac{dt}{d\tau} = \alpha h \tag{2.55}$$

where α is a constant of integration. Equation (2.53b) implies with a further constant of integration

$$r^2 \frac{d\varphi}{dt}\frac{dt}{d\tau} = \beta g \ . \tag{2.56}$$

The last two relations give

$$r^2 \frac{d\varphi}{dt} = \frac{\beta}{\alpha}\frac{g}{h} \tag{2.57}$$

The equations (2.55) and (2.54) yield

$$\left(1 - \frac{1}{\alpha^2 h}\right)\frac{c^2}{h} = \frac{1}{f}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{g}\left(\frac{d\varphi}{dt}\right)^2 . \tag{2.58}$$

Relation (2.57) corresponds to the second Kepler law. The equations (2.58) can be written

$$\left(\frac{dr}{dt}\right)^2 + r^2 \frac{f}{g}\left(\frac{d\varphi}{dt}\right)^2 = c^2\left(1 - \frac{1}{\alpha^2 h}\right)\frac{f}{h} . \tag{2.59}$$

Inserting (2.57) into equation (2.59) we get

$$\left(\frac{dr}{dt}\right)^2 = -\frac{1}{r^2}\left(\frac{\beta}{\alpha}\right)^2 \frac{fg}{h^2} + c^2\left(1 - \frac{1}{\alpha^2 h}\right)\frac{f}{h} . \tag{2.60}$$

The equation (2.60) is a differential equation of first order for $r(t)$. Knowing the solution of (2.60) we have a first order differential equation (2.57) for calculating $\varphi(t)$. These two functions describe the motion of the test particle in the spherically symmetric gravitational field. We will now give the differential equation which describes the trajectory of the test body. We eliminate the time t in the equations (2.57) and (2.59). Furthermore, we put

$$\rho = 1/r \, . \tag{2.61}$$

It follows

$$\frac{d\varphi}{dt}=\frac{\beta}{\alpha}\rho^2\frac{g}{h}$$

and

$$\left(\frac{d\rho}{dt}\right)^2=\rho^4\left(-\left(\frac{\beta}{\alpha}\right)^2\rho^2\frac{fg}{h^2}+c^2\left(1-\frac{1}{\alpha^2 h}\right)\frac{f}{h}\right).$$

The last two equations give

$$\left(\frac{d\rho}{d\varphi}\right)^2=-\rho^2\frac{f}{g}+c^2\left(\frac{\alpha}{\beta}\right)^2\left(h-\frac{1}{\alpha^2}\right)\frac{f}{g^2}\,. \tag{2.62}$$

The differential equation (2.62) describes the inverse ρ of the distance r as a function of the angle φ .

2.6 Redshift

In this sub-chapter the redshift of spectral lines in the gravitational field is studied. It follows by virtue of (1.8) for an atom at rest in the gravitational field the following relation between proper -time and absolute time

$$d\tau=\left(-g_{44}\right)^{1/2}dt=dt/\left(h\left(r\right)\right)^{1/2} \tag{2.63}$$

where (2.4b) is used. This relation gives for the frequency $\nu_e\left(r\right)$ of light emitted from an atom in the gravitational field at distance r from the centre of the body

$$\nu_e\left(r\right)=\nu_0/\left(h\left(r\right)\right)^{1/2} \tag{2.64a}$$

where ν_0 is the frequency of light emitted from the same atom at infinity, i.e. neglecting gravitation. By virtue of Planck's law $E=h\nu$ where h is the Planck constant we get for the emitted energy

$$E\left(r\right)=E_0/\left(h\left(r\right)\right)^{1/2}. \tag{2.64b}$$

This relation follows also by the definition of the energy

$$E \mathrel{\widehat{=}} -g_{4k} \frac{dx^k}{d\tau} \tag{2.65}$$

and the use of (2.4b) and (2.63). Let us assume that the atom at distance r_1 emits light which moves in the gravitational field to the distance r_2. By virtue of (1.30) the energy of light is not changing in the stationary, gravitational field, i.e. the energy (resp. frequency) of light received at r_2 is

$$\nu_r(r_1) = \nu_0 / \left(h(r_1)\right)^{1/2}. \tag{2.66}$$

Light emitted from the same atom at distance r_2 has the frequency

$$\nu_e(r_2) = \nu_0 / \left(h(r_2)\right)^{1/2}. \tag{2.67}$$

Hence the last two relations imply

$$\nu_e(r_2) / \nu_r(r_1) = \left(h(r_1) / h(r_2)\right)^{1/2}. \tag{2.68}$$

The redshift z is then given by

$$z = \frac{\lambda_r}{\lambda_e} - 1 = \frac{\nu_e(r_2)}{\nu_r(r_1)} - 1 = \left(\frac{h(r_1)}{h(r_2)}\right)^{1/2} - 1. \tag{2.69}$$

By virtue of (2.39c) we get to first order approximation

$$z \approx \frac{K}{r_1} - \frac{K}{r_2} \approx \frac{kM_g}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{2.70}$$

i.e. light emitted at r_1 and received at $r_2 > r_1$ gives a redshift z stated by (2.70) to the first order accuracy in agreement with the result of general relativity.

The result (2.70) is by the authors of article $[Pau\ 65]$ experimentally verified in the gravitational field of the Earth with an altitude of $20m$ by the use of the Mössbauer-effect to an accuracy of 1%.

2.7 Deflection of Light

We consider a light ray coming from (r_1, φ_1), passing the nearest point $(r_0, 0)$ to the centre of the body and then moving to the observer at (r_2, φ_2). The equations which describe the motion of this light ray are given in the sub-chapter 2.6. We start from the differential equation (2.62). For the nearest distance r_0 of the light ray to the centre of the body we have

$$\left(\frac{d\rho}{d\varphi}\right)_{\varphi=0} = 0$$

implying by the use of (2.62) and (2.39) to first order approximation in K

$$\left(\frac{\alpha}{\beta}\right)^2 \approx \frac{1}{(r_0 c^2)}\left(1 - 4\frac{K}{r_0} + \frac{1}{\alpha^2}\right). \tag{2.71a}$$

Furthermore, we get for a light ray $d\tau = 0$ by virtue of (2.55)

$$\frac{1}{\alpha} = 0. \tag{2.71b}$$

Substituting the last two relations into equation (2.62) we receive to the first order approximation

$$\left(\frac{d\rho}{d\varphi}\right)^2 = -\rho^2 + \frac{1}{r_0^{\,2}}\left(1 - 4\frac{K}{r_0} + 4K\rho\right). \tag{2.72}$$

The solution of this differential equation with the initial condition $\rho(0) = \rho_0 = 1/r_0$ can be given analytically. We have

$$\varphi = -\int_{\rho_0}^{\rho}\left(-\rho^2 + 4\frac{K}{r_0^{\,2}}\rho + \frac{1}{r_0^{\,2}}\left(1 - 4\frac{K}{r_0}\right)\right)^{-1/2} d\rho$$

Elementary integration and (2.61) give

$$r = r_0 \Big/ \left(2\frac{K}{r_0} + \left(1 - 2\frac{K}{r_0}\right)\cos\varphi\right). \tag{2.73}$$

Inserting the starting point and the end point of the light ray we have for $i=1,2$

$$r_i = r_0 \Big/ \left(2\frac{K}{r_0} + \left(1 - 2\frac{K}{r_0} \right) \cos\varphi_i \right). \tag{2.74}$$

Put for $i=1,2$

$$\varphi_i = \pm\left(\frac{\pi}{2} + \psi_i \right) \tag{2.75a}$$

where the upper (lower) sign stands for $i=1$ $(i=2)$ then we get from (2.74) to first order in K

$$\psi_i \approx 2\frac{K}{r_0} - \frac{r_0}{r_i}. \tag{2.75b}$$

Let γ_i be the angle between the tangent at the light curve in the point (r_i, φ_i) and the x^1 -axis we have

$$ctg\gamma_i = \left(\frac{1}{r_i}\frac{dr_i}{d\varphi_i}\cos(\varphi_i) - \sin\varphi_i \right) \Big/ \left(\frac{1}{r_i}\frac{dr_i}{d\varphi_i}\sin\varphi_i + \cos\varphi_i \right).$$

We have by virtue of (2.71) with (2.61)

$$\left(\frac{dr_i}{d\varphi_i} \right)^2 = r_i^2 \left(-1 + \left(\frac{r_i}{r_0} \right)^2 \left(1 - 4\frac{K}{r_0} + 4\frac{K}{r_i} \right) \right).$$

The last two relations together with (2.75) imply by elementary computations

$$ctg\gamma_i \approx \mp 2\frac{K}{r_0}.$$

The deflection of light is given by $\Delta\gamma = \gamma_1 - \gamma_2$. Hence, we have

$$\begin{aligned}\Delta\gamma &\approx tg(\Delta\gamma) = (tg\gamma_1 - tg\gamma_2)/(1 + tg\gamma_1 tg\gamma_2)\\ &\approx ctg\gamma_2 - ctg\gamma_1\\ &\approx 4\frac{K}{r_0}.\end{aligned} \tag{2.76}$$

The formula (2.76) gives the deflection of light and it is identical with the result of general relativity to the studied approximation.

2.8 Perihelion Shift

We consider now a test particle in the orbit of a spherically symmetric body with velocity

$$|v|^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\varphi}{dt}\right)^2 \ll c^2.$$

Hence, we get from (2.58) and (2.39) to first order approximation to the accuracy of $O\left(\frac{1}{c^2}\right)$:

$$\frac{1}{\alpha^2} \approx 1 + 2\frac{K}{r} - \left(\frac{|v|}{c}\right)^2 \approx 1 - 2\frac{E}{M_g c^2}. \tag{2.77}$$

Here, the conservation law of energy of the test particle in the gravitational field is used to Newtonian accuracy and E is the classical energy satisfying

$$E \ll M_g c^2. \tag{2.78}$$

We get from (2.62) by the use of (2.77), (2.78) and (2.39) to second order in K

$$\left(\frac{d\rho}{d\varphi}\right)^2 = -\rho^2 + c^2\left(\frac{\alpha}{\beta}\right)^2\left(\frac{E}{M_g c^2} + 2K\rho + 6(K\rho)^2\right). \tag{2.79}$$

Put

$$\alpha_1 = 1 - 6(Kc)^2\left(\frac{\alpha}{\beta}\right)^2,\ \alpha_2 = Kc^2\left(\frac{\alpha}{\beta}\right)^2/\alpha_1, \alpha_3 = \frac{E}{M_g}\left(\frac{\alpha}{\beta}\right)^2/\alpha_1. \tag{2.80}$$

The analytic solution of (2.79) with the initial condition $\rho(\varphi_0) = \rho_0$ has the form

$$\rho = \alpha_2 + (\alpha_2^2 + \alpha_3)^{1/2} sin\left\{\alpha_1^{1/2}(\varphi - \varphi_0) + arcsin\left(\frac{\rho_0 - \alpha_2}{(\alpha_2^2 + \alpha_3)^{1/2}}\right)\right\}. \quad (2.81)$$

The solution is an elliptic curve, i.e., there exist two values $\rho_1 > \rho_2 > 0$ such that the right hand side of (2.79). Hence,

$$(\rho_1 - \rho)(\rho - \rho_2) = -\rho^2 + 2\alpha_2\rho + \alpha_3 = 0.$$

This yields

$$\rho_1 + \rho_2 = 2\alpha_2, \ \rho_1\rho_2 = \alpha_3. \quad (2.82)$$

Equation (2.81) gives for a full period the angle

$$\Delta\varphi = \frac{2\pi}{\alpha_1^{1/2}} \approx 2\pi\left(1 + 3K^2c^2\left(\frac{\alpha}{\beta}\right)^2\right).$$

Therefore, we get a perihelion shift

$$\Delta\psi = \Delta\varphi - 2\pi = 6\pi K^2c^2\left(\frac{\alpha}{\beta}\right)^2 \quad (2.83)$$

in the direction of the motion of the test particle.

An elliptic curve with the semi-major axis $a$and the eccentricity $e$satisfies

$$\frac{1}{\rho_1} = a(1 - e), \ \ \frac{1}{\rho_2} = a(1 + e).$$

It follows by (2.82)

$$\frac{2}{a(1-e^2)} = \frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \rho_1 + \rho_2 = 2\alpha_2 \approx 2Kc^2\left(\frac{\alpha}{\beta}\right)^2.$$

Inserting this relation into (2.83) we have

$$\psi = 6\pi\frac{kM_g}{c^2a(1-e^2)}. \quad (2.84)$$

Hence, we get for the perihelion shift of a test particle in a spherically symmetric gravitational field the same result as by Einstein's general theory of relativity.

2.9 Radar Time Delay

We consider a light ray starting from an observer at (r_2, φ_2), passing the spherically symmetric body at $(r_0, 0)$ and reflected at a body with coordinates (r_1, φ_1) and then travelling back to the observer on the same way. We will calculate the needed time and compare it with time when there is no gravitational field.

We start from equation (2.60) for a light ray, i.e., the relations (2.71) hold. Hence, it follows to first order in K

$$\left(\frac{dr}{dt}\right)^2 = -\left(\frac{r_0}{r}\right)^2 c^2 \left(1 + 4\frac{K}{r_0}\right)\frac{fg}{h^2} + c^2\frac{f}{h}.$$

Inserting (2.39) we get

$$\left(\frac{dr}{dt}\right)^2 = -\left(\frac{r_0}{r}\right)^2 c^2 \left(1 + 4\frac{K}{r_0}\right)/\left(1 + 8\frac{K}{r}\right) + c^2\left(1 - 4\frac{K}{r}\right).$$

Therefore, the time for the propagation of a radio signal from $(r_0, 0)$ to $(r_i\,, \varphi_i)$ is

$$t(r_{0\,,} r_i) \approx \frac{1}{c}\int_{r_0}^{r_i} r\left(1 + 4\frac{K}{r}\right)/S(r)dr$$

where

$$S(r) = \left(r^2\left(1 + 4\frac{K}{r}\right) - r_0^2\left(1 + 4\frac{K}{r_0}\right)\right)^{1/2}.$$

Elementary integration gives

$$t(r_0\,, r_i) = \frac{1}{c}\left\{S(r_i) + 2K\,ln\left(\left(r_i + 2K + S(r_i)\right)/(r_0 + 2K)\right)\right\}.$$

We get to first order in K

$$t(r_0, r_i) = \frac{1}{c}\left\{\left(r_i^2 - r_0^2\right)^{1/2} + 2K\left(\frac{r_i - r_0}{r_i + r_0}\right)^{1/2}\right.$$
$$\left.+2K\,ln\left(\left(r_i + \left(r_i^2 - r_0^2\right)^{1/2}\right)/r_0\right)\right\}$$
$$\approx \frac{1}{c}\left(r_i - \frac{1}{2}\frac{r_0^2}{r_i} + 2K + 2K\,ln\frac{2r_i}{r_0}\right).$$

The time of propagation from (r_1, φ_1) to (r_2, φ_2) is

$$t(r_1, r_2) = t(r_0, r_1) + t(r_0, r_2)$$
$$\approx \frac{1}{c}\left(r_1 + r_2 - \frac{1}{2}\left(\frac{r_0^2}{r_1} - \frac{r_0^2}{r_2}\right) + 2K\left(2 + ln\frac{4r_1 r_2}{r_0^2}\right)\right). \tag{2.86}$$

The Euclidean distance between $(r_1, \varphi_1)$and (r_2, φ_2) is

$$R = \{(r_1 cos\varphi_1 - r_2 cos\varphi_2)^2 + (r_1 sin\varphi_1 - r_2 sin\varphi_2)^2\}^{1/2}$$
$$\approx r_1 + r_2 - \frac{1}{2}\frac{r_1 r_2}{r_1 + r_2}(\psi_1 + \psi_2)^2$$

where (2.75a) is used. Inserting (2.75b) it follows to first order in K

$$R \approx r_1 + r_2 - \frac{1}{2}\frac{r_0^2}{r_1 + r_2}\left(\frac{r_2}{r_1} + \frac{r_1}{r_2}\right) - \frac{r_0^2}{r_1 + r_2} + 4K.$$

Hence, we get for the time delay Δt of the radio signal from (r_1, φ_1) to (r_2, φ_2) and back

$$\Delta t = 2\left(t(r_1, r_2) - \frac{R}{c}\right) \approx 4K\, ln\frac{r_1 r_2}{r_0^2}. \tag{2.87}$$

Formula (2.87) is identical with the corresponding result of general relativity in the case when harmonic coordinates are used whereas when Schwarzschild coordinates are considered additional expressions appear (see e.g. [*Wei* 72], [*Log* 86]) . In the theory of gravitation in flat space-time the distance is always the Euclidean one whereas in Einstein's general theory of relativity we have a non-Euclidean geometry implying the mentioned difficulty. Experimental results confirm the result (2.87) to high accuracy (see e.g. [*Sha* 71]).

These results about static spherically symmetric stars with the aid of theory of gravitation in flat space-time can be found in the articles [*Pet* 82, *Pet* 88].

Summarizing, the results of flat space-time theory of gravitation for static spherically symmetric stars agree with the corresponding ones of general relativity to high accuracy by virtue of weak gravitational fields.

2.10 Neutron Stars

To calculate neutron stars we have to solve the differential equations (2.10) and (2.12) together with an equation of state (2.13). The boundary conditions are given by (2.14). The boundary r_0 of the neutron star follows from (2.15) and the mass is given by (2.19) together with (2.16). This problem seems to be not solvable analytically. Numerical methods must be used. The details of the numerical computations can be found in the paper $[Sta\ 84]$and only the results will be given. Several equations of state are considered. For $\rho(r) \leq 5 \cdot 10^{14}\ g/cm^3$ the table of $[Bay\ 71]$ is used and then for $\rho(r) > 5 \cdot 10^{14}\ g/cm^3$ the equations of state are continued by the tables of several authors The results of the flat space-time theory of gravitation are given in the following tables where the author of the continued table is stated.

Table 1. $[Bet\ 74]$

$\rho(0) \cdot 10^{-15}$ *Geben Sie hier eine Formel ein.* g/cm ³	$p(0) \cdot 10^{-1}$ g/cm ³	$f(r_0)$	$g(r_0)$	$h(r_0)$	$M_g/M_\odot$	r_0 km
0.859	0.085	0.765	0.772	1.32	1.05	10.62
2.010	0.547	0.546	0.573	1.90	2.35	10.33
3.160	1.268	0.474	0.513	2.23	2.69	9.58
5.350	3.071	0.426	0.475	2.51	2.77	8.64

Hence, we get with the equation of state of $[Bet\ 74]$ a maximal mass of $2.77M_\odot$ with a radius of $8.64\ km$ and a central density of $5.350 \cdot 10^{15}\ g/cm^3$.

Table 2. $[Wal\ 74]$

$\varrho(0) \cdot 10^{-15}$ g/cm ³	$p(0) \cdot 10^{-15}$ g/cm ³	$f(r_0)$	$g(r_0)$	$h(r_0)$	$M_g/M_\odot$	r_0 km
1.149	0.315	0.526	0.553	1.99	2.88	12.03
2.132	0.974	4.425	0.469	2.54	3.61	11.33
3.060	1.651	0.405	0.455	2.69	3.64	10.82
4.547	2.785	0.399	0.452	2.73	3.53	10.36
8.360	5.829	0.401	0.455	2.69	3.40	10.04

This equation of state gives a maximal mass of a neutron star of $3.64M_\odot$ with a radius of 10.82 km and a central density of matter of $3.06 \cdot 10^{15}\ \frac{g}{cm^3}$.

In the paper $[Hae\ 81]$ several equations of state are studied and the maximal mass of neutron stars is calculated by the use of Einstein's general theory of relativity. Here, we will give for two equations of state the maximal mass, the radius and the density of matter in the centre of the star by flat space-time theory of gravitation. In brackets the corresponding values of general relativity are stated.

$$M = 4.14 M_\odot (= 2.5 M_\odot),\, r_0 = 12.06\, km\ (= 12.1\, km),$$

$$\rho(0) = 2.664 \cdot 10^{15}\, g/cm^3\ (= 2.66 \cdot 10^{15}\, g/cm^3).$$

$$M = 5.13 M_\odot (= 3.1 M_\odot),\, r_0 = 14.83\, km\ (= 12.8\, km),$$

$$\rho(0) = 1.502 \cdot 10^{15}\, g/cm^3.$$

We see that although the radius of the neutron star has in both theories about the same value but the maximal mass can be greater in flat space-time theory of gravitation than that resulting by the use of general relativity.

At last we will calculate neutron stars with a stiff equation of state, i.e.

$$p = \rho - \rho_i + p_i \quad (i = 1,2)$$

where ρ_i and p_i are taken from the table [Bay 71] with

$$\rho_1 = 5.09 \cdot 10^{14}\, g/cm^3, \quad p_1 = 8.22 \cdot 10^{12}\, g/cm^3,$$

$$\rho_2 = 2.00 \cdot 10^{14}\, g/cm^3, \quad p_2 = 1.44 \cdot 10^{12}\, g/cm^3.$$

For $\rho(r) \leq \rho_i$ the equation of [Bay 71] is used again. We get the maximal mass, the approximate radius and the central density of matter

$$M_1 = 5.09 \cdot M_\odot,\, r_{10} = 13.42\, km,\ \ \rho_1(0) = 1.54 \cdot 10^{15}\, g/cm^3.$$

$$M_2 = 8.32 \cdot M_\odot,\, r_{20} = 21.06\, km, \rho_2(0) = 0.57 \cdot 10^{15}\, g/cm^3.$$

Again we remark that the maximal mass of a neutron star can be greater than that received by general relativity. The maximal mass of a neutron star calculated by Einstein's theory with a stiff equation of state is $3.2 \cdot M_\odot$[Rho 74].

Summarizing, the mass of any star estimated by observations may suggest a black hole for this star by general relativity whereas the star can be a neutron star by the use of gravitation in flat space-time.

Details about the numerical calculations and further results on neutron stars can be found in the paper [Sta 84]. Results on neutron stars based on Einstein's theory can be found e.g. in the books [Dem 85] and [Sha 83]. Static neutron stars which have the form of a geoid are numerically computed and can be found in [Neu 87] based on the theory of gravitation in flat space-time.

Chapter 3

Non-Static Spherically Symmetry

In this chapter the theory of gravitation in flat space-time is applied to non-static, spherically symmetric bodies. The results of this chapter are contained in the article [$Pet\ 92b$].

3.1 The Field Equations

The line-element is given by the metric (2.2). The proper time can be written

$$(cd\tau)^2 = -A(r,t)(dr)^2 - B(r,t)r^2((d\vartheta)^2 + \sin^2\theta(d\varphi)^2) \\ +C(r,t)(dct)^2 - 2D(r,t)drdct \tag{3.1}$$

By a transformation of time

$$ct = F(r, c\tilde{t}) \tag{3.2}$$

we can eliminate the expression with $drdct$.With the notation $(\tilde{x}^i) = (r, \vartheta, \varphi, c\tilde{t})$ the line-element can now be written in the form

$$(ds)^2 = -\eta_{ij}d\tilde{x}^i d\tilde{x}^j \tag{3.3a}$$

where

$$\eta_{11} = 1 - (\frac{\partial F}{\partial r})^2, \eta_{22} = r^2, \eta_{33} = r^2 \sin^2\theta, \eta_{44} = (\frac{\partial F}{\partial c\tilde{t}})^2 \\ \eta_{14} = \eta_{41} = \frac{\partial F}{\partial r}\frac{\partial F}{\partial c\tilde{t}}, \eta_{ij} = 0(else) \tag{3.3b}$$

The proper time is now given by

$$(cd\tau)^2 = -g_{ij}d\tilde{x}^i d\tilde{x}^j \tag{3.4a}$$

where

$$g_{11} = 1/f(r,c\tilde{t}), g_{22} = r^2/g(r,c\tilde{t}) \\ g_{33} = r^2\sin^2\vartheta/g(r,c\tilde{t}), \\ g_{44} = -1/h(r,c\tilde{t}), g_{ij} = 0(i \neq j) \tag{3.4b}$$

with new functions $f(r, c\tilde{t}),\ g(r, c\tilde{t})\ and\ h(r, c\tilde{t})$. The energy-momentum tensor of matter described by perfect fluid (1.28) where ρ, p and (u^i) are functions of r and $\tilde{t}$. We have by virtue of spherical symmetry for a collapsing body

$$u^2 = u^3 = 0. \tag{3.5}$$

It follows with $u^i = \frac{d\tilde{x}^i}{d\tau}$ and

$$-g_{ij}u^i u^j = c^2$$

by virtue of (3.5) and (3.4b)

$$u^4 = c\left[h\left(\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)\right]^{1/2}. \tag{3.6}$$

Hence, we consider a non-static spherically symmetric body with only a radial velocity. The energy-momentum tensor of matter (1.28) has by virtue of (3.4), (3.5) and (3.6) the form

$$\begin{aligned} T(M)^i_j &= (\rho + p)\frac{1}{f}(u^1)^2 + pc^2, (i = j = 1) \\ &= \ pc^2, (i = j = 2,3) \\ &= -(\rho + p)c^2\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right) + pc^2, (i = j = 4) \\ &= -(\rho + p)cu^1\left(\frac{1}{h}\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)^{1/2}, (i = 1, j = 4) \\ &= (\rho + p)cu^1\frac{1}{f}\left(h\left(1 + \frac{1}{f}\left(\frac{u^1}{c}\right)^2\right)\right)^{1/2}, (i = 4, j = 1) \\ &= 0, \quad (i \neq j) \end{aligned} \tag{3.7}$$

Put for any function $\beta(r, \tilde{t})$ define

$$\beta_{(1)} = \partial\beta/\partial r, \beta_{(4)} = \partial\,\beta/\partial(c\tilde{t}), \beta_{(14)} = \beta_{(41)} = \partial^2\beta/\partial r\partial\,(c\tilde{t}).$$

For $i = 1, 4$ put

$$L_{ij}=\frac{f_{(i)}}{f}\frac{f_{(j)}}{f}+2\frac{g_{(i)}}{g}\frac{g_{(j)}}{g}+\frac{\left(F_{(4)}^{2}h\right)_{(i)}}{F_{(4)}^{2}h}\frac{\left(F_{(4)}^{2}h\right)_{(j)}}{F_{(4)}^{2}h}$$
$$-\frac{1}{2}\left(\frac{f_{(i)}}{f}+2\frac{g_{(i)}}{g}+\frac{\left(F_{(4)}^{2}h\right)_{(i)}}{F_{(4)}^{2}h}\right)\left(\frac{f_{(j)}}{f}+2\frac{g_{(j)}}{g}+\frac{\left(F_{(4)}^{2}h\right)_{(j)}}{F_{(4)}^{2}h}\right) \tag{3.8}$$

The energy-momentum tensor (1.35) of the gravitational field can be written by virtue of (3.4b) and (3.3b)

$$T(G)_j^i=\frac{1}{8\kappa F_{(4)}g(fh)^{1/2}}$$
$$\times\left\{\frac{f}{2}L_{11}+\frac{h}{2}L_{44}-\frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2-f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2-\frac{2}{r^2}g\left(\frac{(f-g)^2}{fg}-\frac{g}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2\right)\right\}\quad \text{(i=j=1)}$$
$$\times\left\{-\frac{f}{2}L_{11}+\frac{h}{2}L_{44}+\frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2-f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2\right\}\ (\mathrm{i}=\mathrm{j}=2,3)$$
$$\times\left\{-\frac{f}{2}L_{11}-\frac{h}{2}L_{44}+\frac{f^2}{h}\left(\frac{F_{(11)}}{F_{(4)}}\right)^2+f\left(\frac{F_{(14)}}{F_{(4)}}\right)^2-\frac{2}{r^2}g\left(\frac{(f-g)^2}{fg}-\frac{g}{h}\left(\frac{F_{(1)}}{F_{(4)}}\right)^2\right)\right\}\quad \text{(i=j=4)} \tag{3.9}$$
$$\times\left\{fL_{14}-2\frac{f^2}{h}\frac{F_{(11)}}{F_{(4)}}\frac{F_{(14)}}{F_{(4)}}\right\}\quad \text{(i=1, j=4)}$$
$$\times\left\{-hL_{14}+2f\frac{F_{(11)}}{F_{(4)}}\frac{F_{(14)}}{F_{(4)}}\right\}\quad \text{(i=4, j=1)}$$
$$\times\{0\}\quad \text{(else)}$$

We get from (1.21b) with (1.10)

$$T(\Lambda)_j^i=-\frac{\Lambda}{2\kappa}\frac{1}{F_{(4)}g(fh)^{1/2}}\delta_j^i. \tag{3.10}$$

Let us define the following differential operators L_1 and L_2 of order two:

$$L_1(y):=\frac{1}{r^2F_{(4)}}\left\{\frac{\partial}{\partial r}\left(r^2\frac{f}{g(fh)^{1/2}}\frac{y_{(1)}}{y}\right)-\frac{\partial}{\partial c\tilde{t}}\left(r^2\frac{h}{g(fh)^{1/2}}\frac{y_{(4)}}{y}\right)\right\} \tag{3.11a}$$

$$L_2(y):=\frac{1}{r^2F_{(4)}}\left\{\frac{\partial}{\partial r}\left(r^2\frac{f^2}{F_{(4)}hg(fh)^{1/2}}y_{(1)}\right)-\frac{\partial}{\partial c\tilde{t}}\left(r^2\frac{f}{F_{(4)}g(fh)^{1/2}}y_{(4)}\right)\right\} \tag{3.11b}$$

Then, the field equations (1.24) with (1.23a) and (1.23c) have by virtue of (3.4b), (3.7), (3.9) and (3.10) the following form

$$\begin{aligned} L_1(f) = \frac{1}{F_{(4)} g (fh)^{1/2}} \left\{ \frac{2}{r^2} \frac{f^2 - g^2}{f} + f \left(\frac{F_{(14)}}{F_{(4)}} \right)^2 - 2 \frac{f^2}{h} \left(\frac{F_{(11)}}{F_{(4)}} \right)^2 + \frac{1}{2} f L_{11} + 2\Lambda \right\} \\ + 2\kappa \left[(\rho - p) c^2 + 2(\rho + p) \frac{1}{f} (u^1)^2 \right] \end{aligned} \tag{3.12a}$$

$$L_1(g) = \frac{1}{F_{(4)} g (fh)^{1/2}} \left\{ - \frac{2}{r^2} \frac{g(f-g)}{f} - \frac{2}{r^2} \frac{g^2}{h} \left(\frac{F_{(1)}}{F_{(4)}} \right)^2 + 2\Lambda \right\} + 2\kappa (\rho - p) c^2 \tag{3.12b}$$

$$\begin{aligned} L_1\left(F_{(4)}^2 h\right) = \frac{1}{F_{(4)} g (fh)^{1/2}} \left\{ \frac{2}{r^2} \frac{g^2}{h} \left(\frac{F_{(1)}}{F_{(4)}} \right)^2 + \frac{f^2}{h} \left(\frac{F_{(11)}}{F_{(4)}} \right)^2 - \frac{1}{2} h L_{44} + 2\Lambda \right\} \\ - 2\kappa \left[(\rho + 3p) c^2 + 2(\varrho + p) \frac{1}{f} (u^1)^2 \right] \end{aligned} \tag{3.12c}$$

$$\begin{aligned} L_2\left(F_{(1)}\right) = \frac{1}{F_{(4)} g (fh)^{1/2}} \left\{ \frac{2}{r^2} \frac{g^2}{h} \frac{F_{(1)}}{F_{(4)}} + f \frac{F_{(14)}}{F_{(4)}} \left(2 \frac{f}{h} \frac{F_{(11)}}{F_{(4)}} - \frac{F_{(44)}}{F_{(4)}} \right) - \frac{1}{2} f L_{14} \right\} \\ + 4\kappa (\rho + p) c u^1 \left(\frac{1}{h} \left(1 + \frac{1}{f} \left(\frac{u^1}{c} \right)^2 \right) \right)^{1/2} . \end{aligned} \tag{3.12d}$$

The field equations (3.12) are four partial differential equations for the four unknown functions f, g, h and F defining by the use of (3.4b) the gravitational potentials (g_{ij}).

3.2 Equations of Motion and Energy-Momentum Conservation

In flat space-time theory of gravitation we have in addition to the field equations the equations of motion (1.29a) and the conservation law (1.25a) of the whole energy-momentum. One of these equations follows by the other one and can be omitted. The equations of motion (1.29a) yield by the use of (3.3b), (3.4b) and (3.7) the two equations

$$\begin{aligned}&\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T(M)_1^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T(M)_1^4\right)\right\}\\&=-\frac{1}{2}\frac{f_{(1)}}{f}T(M)_1^1+\left(\frac{2}{r}-\frac{g_{(1)}}{g}\right)T(M)_2^2-\frac{h_{(1)}}{h}T(M)_4^4\end{aligned}\tag{3.13a}$$

$$\begin{aligned}&\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T(M)_4^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T(M)_4^4\right)\right\}\\&=-\frac{1}{2}\frac{f_{(4)}}{f}T(M)_1^1-\frac{g_{(4)}}{g}T(M)_2^2-\frac{1}{2}\frac{h_{(4)}}{h}T(M)_4^4\end{aligned}\tag{3.13b}$$

Equation (1.25) implies the following two equations for the whole energy-momentum

$$\frac{1}{r^2F_4}\left\{\frac{\partial}{\partial r}\left(r^2F_{(4)}T_1^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}T_1^4\right)\right\}-\frac{2}{r}T_2^2-\frac{F_{(11)}}{F_{(4)}}T_4^1-\frac{F_{(14)}}{F_{(4)}}T_4^4\tag{3.14a}$$

$$\frac{\partial}{\partial r}(r^2T_4^1)+\frac{\partial}{\partial c\tilde{t}}(r^2T_4^4)=0.\tag{3.14b}$$

The equations (3.12), (3.13) and (3.14) describe a spherically symmetric collapsing body where one of the equations (3.13) or (3.14) can be omitted. Furthermore, $\Lambda=0$ is assumed for a star by virtue of its smallness. In general one replaces ρ by $\rho+\rho\Pi$ where Π denotes the specific internal energy and one adds the conservation law of matter (1.29b) and an equation of state of the form

$$p=p(\rho,\Pi).\tag{3.15}$$

Hence, we have eight unknown functions f, g, h, ρ, p Π and u^1 (u^4 follows by (3.6)) depending on r and $\tilde{t}$ and eight independent equations (3.12) (four equations), (3.13) or (3.14) (two equations), (1.29b) (one equation), and (3.15) (one equation).

A solution of these equations for a collapsing star is at present time not known, also numerical solutions are not computed. Therefore, it is an open question whether black holes exist or not by the use of this flat space-time theory of gravitation.

The corresponding equations by Einstein's general theory of relativity are stated e.g. in the papers [*May* 66] and [*Mis* 64]. They are simpler than the above ones because Einstein's theory allows to reduce the number of unknown functions by suitable transformations.

Chapter 4

Rotating Stars

In this chapter the theory of gravitation in flat space-time of chapter I will be applied to rotating stars. Rotating neutron stars (pulsars) are numerically computed. All the results of this chapter are contained in the work of [Kus 88] where additional details can be found.

4.1 Field Equations

The line-element is again given by (1.1) with (2.2).The gravitational potentials are:

$$\begin{aligned} &g_{11}=\frac{1}{f},\ g_{22}=\frac{r^2}{g},\ g_{33}=\frac{r^2\sin^2\vartheta}{d},\ g_{44}=-\frac{1}{h}, \\ &g_{12}=g_{21}=ar,\ g_{34}=g_{43}=br\sin\vartheta,\ g_{ij}=0(else) \end{aligned} \tag{4.1}$$

Where the six functions f, g, d, h, a, b depend on r and ϑ. Put

$$\Omega_1=\frac{1}{fg}-a^2,\ \ \Omega_2=\frac{1}{dh}+b^2 \tag{4.2}$$

Then, we get

$$(-G)^{1/2}=r^2 sin\vartheta(\Omega_1\Omega_2)^{1/2}. \tag{4.3}$$

The energy-momentum tensor of matter is given by perfect fluid (1.28) where ρ, p and (u^i) are functions of r and ϑ. Let us assume that the star is rotating with constant angular velocity about an axis then the four-velocity is

$$\left(u^i\right)=\left(\frac{dr}{d\tau},\frac{d\vartheta}{d\tau},\frac{d\varphi}{d\tau},c\,\frac{dt}{d\tau}\right)=(0,0,\omega,c)\,\frac{dt}{d\tau}. \tag{4.4}$$

Put

$$\begin{aligned} &z_1=-\omega^2r^2\sin^2\vartheta\frac{1}{d}-cr\omega b\sin\vartheta \\ &z_2=\frac{c^2}{h}-cr\omega b\sin\vartheta \end{aligned} \tag{4.5}$$

Then, we get from relation (1.8)

$$\frac{d\tau}{dt}=\frac{1}{c}(z_1+z_2)^{1/2}. \tag{4.6}$$

Hence, we receive from (4.4) the four-velocity

$$u^1=u^2=0, u^3=\omega c(z_1+z_2)^{-1/2}, u^4=c^2(z_1+z_2)^{-1/2}. \tag{4.7}$$

Therefore, the energy-momentum tensor of matter (1.28) has by virtue of (4.1) and (4.7) the form

$$\begin{aligned} T(M)^i_j &= pc^2 \ (i=j=1,2) \\ &= pc^2 - (\rho+p)c^2\, z_1/(z_1+z_2)\ (i=j=3) \\ &= pc^2 - (\rho+p)c^2\, z_2/(z_1+z_2)\ (i=j=4) \\ &= -(\rho+p)\omega c\, z_2/(z_1+z_2)\ (i=3, j=4) \\ &= -(\rho+p)(c^3/\omega)\, z_1/(z_1+z_2)\ (i=4, j=3) \\ &= 0.\ (else) \end{aligned} \tag{4.8}$$

The formal representation of the energy-momentum tensor of the gravitational field with the aid of the potential functions f, g, d, h, a, b will now be stated. With the aid of complicated tensors D_{ij} (i, j=1,2) with $D_{12} = D_{21}$ which contain quadratic expressions of first order derivatives of the potentials let us define new tensors

$$H^i_j = g^{ik} D_{jk}. \tag{4.9}$$

Then, the components of the energy-momentum tensor of the gravitational field needed subsequently for the field equations are

$$\begin{aligned} T(G)^i_j &= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(H^1_1 - H^2_2 - H^3_3) \quad \text{(i=j=1)} \\ &= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(-H^1_1 + H^2_2 - H^3_3) \quad \text{(i=j=2)} \\ &= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(-H^1_1 - H^2_2 + H^3_3) \quad \text{(i=j=3)} \\ &= -\frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(H^1_1 - H^2_2 - H^3_3) \quad \text{(i=j=4)} \\ &= \frac{(\Omega_1\Omega_2)^{1/2}}{8\kappa} H^1_2 \quad \text{(i=1, j=2)} \end{aligned} \tag{4.10}$$

To get the field equations we put for any function $\beta(r, \vartheta)$:

$$\beta_{(1)} = \frac{\partial\beta}{\partial r},\ \ \beta_2 = \frac{\partial\beta}{\partial\vartheta},\ \ \beta_{12} = \frac{\partial^2\beta}{\partial r\partial\vartheta} = \frac{\partial^2\beta}{\partial\vartheta\partial r}. \tag{4.11}$$

Furthermore, we define for $i = 1,2$

$$F_i = \frac{1}{f}\left(\frac{1}{g\Omega_1}\right)_{(i)} - a\left(\frac{a}{\Omega_1}\right)_{(i)} + \frac{a}{\Omega_1}\left(\frac{1}{f} + \frac{1}{g}\right)\delta_{i2}$$

$$G_i = \frac{1}{g}\left(\frac{1}{f\Omega_1}\right)_{(i)} - a\left(\frac{a}{\Omega_1}\right)_{(i)} - \frac{a}{\Omega_1}\left(\frac{1}{f} + \frac{1}{g}\right)\delta_{i2}$$

$$D_i = \frac{1}{d}\left(\frac{1}{h\Omega_2}\right)_{(i)} + b\left(\frac{b}{\Omega_2}\right)_{(i)}$$

$$H_i = \frac{1}{h}\left(\frac{1}{d\Omega_2}\right)_{i} + b\left(\frac{b}{\Omega_2}\right)_{(i)} \tag{4.12}$$

$$A_i = a\left(\frac{1}{g\Omega_1}\right)_{(i)} - \frac{1}{g}\left(\frac{a}{\Omega_1}\right)_{(i)} + \left(-1 + \frac{1+a^2g^2}{\Omega_1 g^2}\right)\delta_{i2}$$

$$B_i = b\left(\frac{1}{h\Omega_2}\right)_{(i)} - \frac{1}{h}\left(\frac{b}{\Omega_2}\right)_{(i)}$$

$$M_i = a\left(\left(\frac{1}{f\Omega_1}\right)_{(i)} + \left(\frac{1}{g\Omega_1}\right)_{(i)}\right) - \left(\frac{a}{\Omega_1}\right)_{(i)}\left(\frac{1}{f} + \frac{1}{g}\right) + \frac{1}{\Omega_1}\left(\frac{1}{g^2} - \frac{1}{f^2}\right)\delta_{i2}$$

$$N_i = \frac{1}{f}\left(\frac{1}{g\Omega_1}\right)_{(i)} - \frac{1}{g}\left(\frac{1}{f\Omega_1}\right)_{(i)} + \frac{2a}{\Omega_1}\left(\frac{1}{f} + \frac{1}{g}\right)\delta_{i2}.$$

In addition put

$$\begin{aligned} y_1 &= \frac{\sin^2\vartheta}{g} - a\sin\vartheta\cos\vartheta,\ y_2 = \frac{\sin^2\vartheta}{f} + a\sin\vartheta\cos\vartheta, \\ y_3 &= \frac{\cos^2\vartheta}{f} - a\sin\cos\vartheta,\ y_4 = \frac{\cos^2\vartheta}{g} + a\sin\vartheta\cos\vartheta, \\ y_5 &= -a\cos^2\vartheta + \frac{\sin\vartheta\cos\vartheta}{g},\ y_6 = a\sin^2\vartheta + \frac{\sin\vartheta\cos\vartheta}{g} \end{aligned} \tag{4.13}$$

The field equations (1.24) with $\Lambda = 0$ give the following system of differential equations:

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}F_1 - \frac{a}{r\Omega_1}F_2\right)\right] \\ &+ \frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}F_1 + \frac{1}{f\Omega_1}F_2\right)\right] \end{aligned} \tag{4.14}$$

$$-(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}M_1+\frac{1}{fr^2\Omega_1}M_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2hsin^2\vartheta\Omega_2}\left(\frac{y_1}{d\Omega_1}-\frac{y_2}{h\Omega_2}\right)$$

$$=2\kappa(\rho-p)c^2+\frac{1}{2}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{12}+\frac{1}{g\Omega_1}D_{11}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}G_1-\frac{a}{r\Omega_1}G_2\right)\right]$$

$$+\frac{1}{r^2sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}G_1+\frac{1}{f\Omega_1}G_2\right)\right]$$

$$+(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}M_1+\frac{1}{fr^2\Omega_1}M_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2hsin^2\vartheta\Omega_2}\left(\frac{y_3}{d\Omega_1}-\frac{y_4}{h\Omega_2}\right)$$

$$=2\kappa(\rho-p)c^2+\frac{1}{2}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{12}+\frac{1}{fr^2\Omega_1}D_{22}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}D_1-\frac{a}{r\Omega_1}D_2\right)\right]$$

$$+\frac{1}{r^2sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}D_1+\frac{1}{f\Omega_1}D_2\right)\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2hsin^2\vartheta\Omega_2}\left[\frac{1}{d\Omega_1}(y_1+y_3)-\frac{1}{h\Omega_2}(y_2+y_4)\right]$$

$$=2\kappa c^2\left[(\rho-p)-2(\rho+p)\frac{z_1}{z_1+z_2}\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2hsin^2\vartheta\Omega_2}\left[\frac{1}{d\Omega_1}(y_1+y_3)+\frac{1}{h\Omega_2}(y_2+y_4)-2\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}H_1-\frac{a}{r\Omega_1}H_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}H_1+\frac{1}{f\Omega_1}H_2\right)\right]$$

$$=2\kappa c^2\left[(\rho-p)-2(\rho+p)\frac{z_2}{z_1+z_2}\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}A_1-\frac{a}{r\Omega_1}A_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}A_1+\frac{1}{f\Omega_1}A_2\right)\right]$$

$$+(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}N_1+\frac{1}{r^2 f\Omega_1}N_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\left(\frac{y_5}{d\Omega_1}-\frac{y_6}{h\Omega_2}\right)$$

$$=\frac{1}{2r}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{22}+\frac{1}{g\Omega_1}D_{12}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}B_1-\frac{a}{r\Omega_1}B_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}B_1+\frac{1}{f\Omega_1}B_2\right)\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\frac{b}{\Omega_1}(y_1+y_3)$$

$$=-4\kappa c\omega r sin\vartheta(\rho+p)\frac{z_2}{z_1+z_2}.$$

Hence, we have six differential equations for the six potential functions f, g, d, h, a, b. It is worth mentioning that the equation (4.14) for the function a does not depend on the matter tensor (4.8) but it cannot be omitted because the equation implies that $a \neq 0$.

4.2 Equations of Motion

We get from the equations (1.29a) by the use of (4.1) and (4.8) the following two differential equations

$$\begin{aligned}\frac{dp}{dr} = &-\frac{1}{2}\left[\frac{1}{fg\Omega_1}\left(\frac{f_{(1)}}{f}+\frac{g_{(1)}}{g}\right)+\frac{1}{dh\Omega_2}\left(\frac{d_{(1)}}{d}+\frac{h_{(1)}}{h}\right)\right]p \\ &+\frac{1}{r}\left[\frac{b(rb)_{(1)}}{\Omega_2}-\frac{a(ra)_1}{\Omega_1}\right]p+\frac{1}{r}\left[\frac{1}{fg\Omega_1}+\frac{1}{dh\Omega_2}-2\right]p \\ &-\frac{1}{2}\frac{(z_1+z_2)_{(1)}}{z_1+z_2}(\rho+p), \\ \frac{dp}{d\vartheta} = &-\frac{1}{2}\left[\frac{1}{fg\Omega_1}\left(\frac{f_{(2)}}{f}+\frac{g_{(2)}}{g}\right)+\frac{1}{dh\Omega_2}\left(\frac{d_{(2)}}{d}+\frac{h_{(2)}}{h}\right)-2\left(\frac{bb_{(2)}}{\Omega_2}-\frac{aa_{(2)}}{\Omega_1}\right)\right]p \\ &-\frac{1}{2}\frac{(z_1+z_2)_{(2)}}{z_1+z_2}(\rho+p).\end{aligned} \tag{4.15}$$

The conservation of the mass (1.29b) is fulfilled by virtue of (4.4).

The boundary of the rotating star is given by the condition

$$\mathrm{p}(r,\vartheta)=0 \tag{4.16}$$

implying that the boundary $r(\vartheta)$ depends on the angle ϑ by virtue of the rotation of the star. Hence, spherical symmetry cannot hold.

It is worth mentioning that we have two equations of motion (4.15) but by virtue of the equations of state of the form (2.13) we have only one function $\rho(r,\vartheta)$. This is connected with the assumption that we consider a rigid body rotating about a fixed axis with constant angular velocity $\frac{d\varphi}{dt}=\omega$. Hence, a perfect fluid for a rigid body rotating about a fixed axis with constant angular velocity ω and a velocity of the form (4.4) does not exist because gravitational forces work. This gives a solution to the paradox of a uniformly rotating disc noted by Ehrenfest and considered as justification for the introduction of non-Euclidean geometry in general relativity theory of Einstein (see e.g.[Mol 72]). It should be mentioned that any transformation of the pseudo-Euclidean geometry conserves the flat space-time metric. Therefore, we have for a body rotating about an axis to introduce additional velocities $\frac{dr}{dt}\neq 0$ and perhaps also $\frac{d\vartheta}{dt}\neq 0$ as functions of r and ϑ which will give new differential equations. A simpler possibility without great changes of the equations is the assumption that ω is a function of r and ϑ.

Rotating stars studied with the aid of general relativity can be found e.g. in the article $[Dem\ 85]$.

4.3 Rotating Neutron Stars

In the following, we consider a rotating neutron star with constant angular velocity ω approximately described by the equations (4.14) and (4.15). The results of this sub-chapter can be found in the work of $[Kus\ 88]$.

To simplify the equations (4.14) and (4.15) all small functions a and b in the non-linear expressions are neglected, i.e. we consider to the lowest order a non-rotating star studied in chapter 2.10 where all functions only depend on r . We put for any function

$$\beta(r,\vartheta) \approx \beta_0(r) + \beta_\varepsilon(r,\vartheta). \tag{4.17}$$

The function $\beta_0(r)$ describes the non-rotating star and β_ε is the small correction implied by the rotation. Then, we have by virtue of sub-chapter 2.10

$$d_0 = g_0, \quad a_0 = b_0 = \omega_0 = 0. \tag{4.18}$$

For the angular velocity it is assumed that the expression

$$\Omega := \frac{K}{c}\omega << 1. \tag{4.19}$$

The transformations

$$p = \left(\frac{-G}{-\eta}\right)^{1/2} \tilde{p}\ , \rho = \left(\frac{-G}{-\eta}\right)^{1/2} \tilde{\rho} \tag{4.20}$$

give new expressions $\tilde{p}$ and $\tilde{\rho}$ for pressure and density. In the following quadratic expressions of a and b are omitted. Then, these approximations imply with the aid of (4.3) and (4.2)

$$p = \frac{\tilde{p}}{(fgdh)^{1/2}}, \quad \rho = \frac{\tilde{\rho}}{(fgdh)^{1/2}}. \tag{4.21}$$

Now, the equations of motion (4.15) can for $i = 1,2$ approximately be written by the use of (4.4) and by neglecting quadratic expressions of the small quantities a and b in the form

$$\tilde{p}_{(i)} = -\frac{1}{2}(\tilde{\rho} + \tilde{p})\frac{(z_1+z_2)_{(i)}}{z_1+z_2}. \tag{4.22}$$

Let us now assume an equation of state

$$\tilde{p} = \tilde{p}(\tilde{\rho}) \tag{4.23a}$$

Therefore, on the above approximations and the assumption (4.23a) with (4.20) the two differential equations (4.22) depend on one another in contrast to the general case (4.15). In the following, we use an equation of state

$$\tilde{p} = C\tilde{\rho}^{1+\gamma}, \gamma = 2/3 \tag{4.23b}$$

with a suitable constant C. Then, the two differential equations (4.22) have a unique solution for $\tilde{\rho}$ which will not be given. By virtue of (4.23b) $\tilde{p}$ can be calculated and we get p and ρ by (4.21). It seems more natural to assume an equation of state for p as function of ρ instead of $\tilde{p}$ as function of $\tilde{\rho}$. Hence, we see that a rigid body can be approximated on some assumptions but it does not really exist. With the approximation of the form (4.17) for the pressure $\tilde{p}$ and $\tilde{\rho}$ we can calculate the pressure p_0 and the density ρ_0 of the non-rotating neutron star and the corresponding approximated values p_ε and ρ_ε which follow from the above equation. Substituting all the expressions into the field equations (4.14) we get to the lowest order three ordinary differential equations for the functions $f_0(r)$, $g_0(r)$ and $h_0(r)$ describing the non-rotating neutron star. In addition, by linearization of the equations (4.14) we get six linear partial differential equations for the approximated functions f_ε, g_ϵ, d_ε, h_ε, a_ε and b_ε depending on r and ϑ. It is worth mentioning that the equations describing the non-rotating neutron star are different from those of sub-chapter 2.10 by virtue of the different equation of state.

Numerical methods are used for the solution of the problem where angular velocities $\omega \in [570, 1034]\frac{1}{sec}$ are considered fulfilling the assumption (4.19) and being observed (see $[Bac\ 82]$). We will give some of these results in the following table where in the column with $\omega = 0$ the results for the non-rotating star are stated. Here, a and b denote the semi-major and semi-minor axes of the flattened rotating body (pulsar) where $a = b$ gives the neutron star.

Table

$\tilde{\rho}(0) \cdot 10^{-15}$ g/cm^3	$\tilde{p}(0) \cdot 10^{-15}$ g/cm^3	$M/M_\odot$	a km	$b\ km$	ω $1/sec$
0.895	0.0850	1.30	14.4	14.4	0
0.895	0.0850	1.31	14.4	14.4	570
0.895	0.0850	1.33	14.4	14.3	1000
0.898	0.0852	1.35	14.6	14.0	2000
0.904	0.0855	1.38	14.6	13.8	3000
0.916	0.0862	1.42	14.91	13.3	4030
1.53	0.317	1.98	14.6	14.6	0
1.54	0.138	2.06	14.8	14.2	1000
2.01	0.547	2.09	14.0	14.0	0
2.03	0.550	2.17	14.2	13.6	1000

The table shows that greater angular velocities and greater densities with greater pressure give greater deviations of the mass and semi-axes. For too small angular velocities and densities there are negligible deviations of mass and of semi-axes from those of non-rotating neutron stars.

Furthermore, it follows by comparison with the results of sub-chapter 2.10 that an equation of state of the form (2.13) gives a greater mass than an equation of state (4.23a) with (4.21).

More details of the approximations and the numerical computations of rotating stars can be found in the work of [Kus 88].

Results about neutron stars based on the theory of general relativity are given by [Har 67] and can also be found in [Dem 85] containing further references.

Chapter 5

Post-Newtonian Approximation

In this chapter post-Newtonian approximation of the gravitational field in flat space-time of a perfect fluid is studied. The conservation laws of energy-momentum and of angular-momentum are derived. The equivalence of the conservation law of energy-momentum and of the equations of motion is shown to the studied accuracy. All the results of post-Newtonian approximation in flat space-time theory of gravitation agree up to the studied accuracy with those of general relativity as studied by Will in his famous book of Will [Wil 81].

5.1 Post-Newtonian Approximation

The study of post-Newtonian approximation of gravitation in flat space-time follows along the considerations of Will. In this sub-chapter we assume a matter tensor of the form

$$T(M)_i^j = \left(\frac{-G}{-\eta}\right)^{1/2} \left\{\left(\rho\left(1+\frac{\Pi}{c^2}\right)+p\right) g_{ik}\frac{dx^k}{d\tau}\frac{dx^j}{d\tau} + pc^2\delta_i^j\right\} \tag{5.1a}$$

where ρ denotes the density of matter, Π is the specific internal energy, p is the isotropic pressure and $\left(\frac{dx^i}{d\tau}\right)$ is the four-velocity. Equation (5.1a) yields by the use of relation (1.8)

$$T(M)_k^k = -\left(\frac{-G}{-\eta}\right)^{1/2} \left(\rho\left(1+\frac{\Pi}{c^2}\right) - 3p\right) c^2. \tag{5.1b}$$

The post-Newtonian approximation is an expansion of the gravitational field in powers of $\frac{1}{c}$. Subsequently, we use the pseudo-Euclidean geometry given by (1.4) and (1.5). Let us start with the Newtonian gravitational potential defined by

$$\Delta U = -4\pi k\rho \tag{5.2a}$$

with the solution

$$U(x,t) = k\int \frac{\rho(x',t)}{|x-x\prime|} dx'^3. \tag{5.2b}$$

Sub-chapter 2.2 implies the approximate tensor

$$\begin{aligned} g_{ij} &= 1 + \frac{2}{c^2}U (i = j = 1,2,3) \\ &= -\left(1 - \frac{2}{c^2}U\right)(i = j = 4) \\ &= 0 (i \neq j) \end{aligned} \tag{5.3a}$$

with the inverse tensor

$$\begin{aligned} g^{ij} &= 1 - \frac{2}{c^2}U (i = j = 1,2,3) \\ &= -\left(1 + \frac{2}{c^2}U\right)(i = j = 4) \\ &= 0 (i \neq j) \end{aligned} \tag{5.3b}$$

Let v denote the velocity of the body, i.e.

$$v = (v^1, v^2, v^3) = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) \tag{5.4}$$

And assume in analogy to Will

$$\left|\frac{v}{c}\right|^2 \sim \frac{1}{c^2}U \sim \frac{p}{\rho} \sim \frac{\Pi}{c^2} \sim O\left(\frac{1}{c^2}\right) \tag{5.5a}$$

and

$$\left|\frac{\partial/\partial t}{\partial/\partial x^i}\right| \sim O(1). \tag{5.5b}$$

The post-Newtonian approximation of gravitation now requires the knowledge of g_{44} to $O\left(\frac{1}{c^4}\right)$, of g_{i4} to $O\left(\frac{1}{c^3}\right)$ and of g_{ij} to $O\left(\frac{1}{c^2}\right)$ (i,j=1,2,3). Hence, we make the ansatz

$$\begin{aligned} g_{ij} &= \left(1 + \frac{2}{c^2}U\right)\delta_{ij} (i, j = 1,2,3) \\ &= -\frac{4}{c^3}V_i (i = 1,2,3; j = 4) \\ &= -\frac{4}{c^3}V_j (i = 4; j = 1,2,3) \\ &= -\left(1 - \frac{2}{c^2}U + \frac{1}{c^4}S\right)(i = j = 4) \end{aligned} \tag{5.6a}$$

with

$$V_i \sim S \sim O(1). \tag{5.6b}$$

The inverse tensor of (5.6a) is

$$\begin{aligned} g^{ij} &= \left(1 - \frac{2}{c^2}U\right)\delta^{ij} (i;\, j = 1, 2, 3) \\ &= -\frac{4}{c^3}V_i (i = 1, 2, 3;\, j = 4) \\ &= -\frac{4}{c^3}V_j (i = 4;\, j = 1, 2, 3) \\ &= -\left(1 + \frac{2}{c^2}U + \frac{1}{c^4}(-S + 4U^2)\right)(i = j = 4) \end{aligned} \tag{5.6c}$$

In addition, we have

$$\left(\frac{-G}{-\eta}\right)^{1/2} \approx 1 + \frac{2}{c^2}U \tag{5.6d}$$

It follows from (1.13) and (1.12) by the use of (5.4) and (5.6)

$$\frac{dt}{d\tau} \approx 1 + \frac{1}{c^2}U + \frac{1}{2}\left|\frac{v}{c}\right|^2. \tag{5.7}$$

We get from (5.1) with the aid of (5.6) and (5.7)

$$\begin{aligned} T(M)^i_j &= \rho v^i v^j + pc^2\delta^i_j \quad (\mathrm{i};\ \mathrm{j} = 1, 2, 3) \\ &= \rho c v^i \left(1 + \frac{\Pi}{c^2} + \frac{6}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right) - \frac{4}{c}\rho V_j \\ &\qquad (\mathrm{i} = 4;\ \mathrm{j} = 1, 2, 3) \\ &= -\rho c v^i \left(1 + \frac{\Pi}{c^2} + \frac{2}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right) \quad (\mathrm{i}=1, 2, 3;\ \mathrm{j}=4) \\ &= -\rho c^2 \left(1 + \frac{\Pi}{c^2} + \frac{2}{c^2}U + \left|\frac{v}{c}\right|^2\right) \quad (\mathrm{i}=\mathrm{j}=4) \end{aligned} \tag{5.8a}$$

to $O(1)$ and $O\left(\frac{1}{c}\right)$ respectively. Furthermore, we get to $O(1)$

$$T(M)^k_k = -\rho c^2 \left(1 + \frac{\Pi}{c^2} + \frac{2}{c^2}U - 3\frac{p}{\rho}\right). \tag{5.8b}$$

We have from (1.21a) and (1.9) by the use of (5.6) the mixed energy-momentum tensor of the gravitational field to the same accuracy as that of matter

$$\begin{aligned} T(G)_j^i &= \frac{1}{8\kappa}\frac{8}{c^4}\left(\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{1}{2}\delta_j^i \sum_{k=1}^3 \frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k}\right) \quad \text{(i; j=1,2,3)} \\ &= -\frac{1}{8\kappa}\frac{8}{c^4}\frac{\partial U}{\partial ct}\frac{\partial U}{\partial x^j} \quad \text{(i=4; j=1, 2, 3)} \\ &= +\frac{1}{8\kappa}\frac{8}{c^4}\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial ct} \quad \text{(i=1, 2, 3; j=4)} \\ &= -\frac{1}{8\kappa}\frac{4}{c^4}\sum_{k=1}^3 \frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} \quad \text{(i=j=4)} \end{aligned} \tag{5.9a}$$

and

$$T(G)_l^l = -\frac{1}{8\kappa}\frac{8}{c^4}\sum_{k=1}^3 \frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k}. \tag{5.9b}$$

We now obtain from (1.24) with the aid of (5.6), (5.8), (5.9) and (5.2) by longer elementary calculations

$$\Delta V_i = -4\pi k \rho v^i \quad \text{(i=1,2,3)} \tag{5.10a}$$

And

$$\Delta S - 4\sum_{k=1}^3 \frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right) + 2\frac{\partial^2 U}{\partial t^2} = 8\pi k\rho\left(\Pi + 2U + 2|v|^2 + 3\frac{pc^2}{\rho}\right). \tag{5.10b}$$

Here, (5.10a) follows with i=1,2,3; j=4 (or i and j exchanged) and equation (5.10b) with i=j=4. The equations (1.23) with i, j=1,2,3 are identically satisfied by virtue of (5.2).The solution of (5.10a) is given by

$$V_i = k\int \frac{\rho' v^{i\prime}}{|x-x'|} d^3x' \quad \text{(i=1,2,3)} \tag{5.11}$$

where $\rho' = \rho(x', t)$ and correspondingly $v^{i'}$.To solve equation (5.10b) we use the identity

$$\Delta U^2 = 2\sum_{k=1}^3 \frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right)$$

and introduce in analogy to Chandrasekhar the super-potential

$$\chi = -k\int \rho'|x-x'|\, d^3x' \tag{5.12a}$$

which satisfies

$$\triangle\, \chi = -2U. \tag{5.12b}$$

Hence, the equation (5.10b) can be rewritten

$$\triangle \left(S - 2U^2 - \frac{\partial^2 \chi}{\partial t^2}\right) = 8\pi k\rho \left(\Pi + 2U + 2|v|^2 + 3\frac{pc^2}{\rho}\right). \tag{5.13}$$

Furthermore, let us put $(see\ |Wil\ 81|)$

$$\begin{aligned} &\phi_1 = k\int \frac{\rho' |v'|^2}{|x - x'|} d^3x', \phi_2 = k\int \frac{\rho' U'}{|x - x'|} d^3x', \\ &\phi_3 = k\int \frac{\rho' \Pi'}{|x - x'|} d^3x', \phi_4 = k\int \frac{p'}{|x - x'|} d^3x'. \end{aligned} \tag{5.14a}$$

and

$$\phi = 2\phi_1 + 2\phi_2 + \phi_3 + 3\phi_4 \tag{5.14b}$$

then, the equation (5.13) has the solution

$$S = 2U^2 + \frac{\partial^2 \chi}{\partial t^2} - 2\phi. \tag{5.15}$$

Hence, the tensors (g_{ij}) and (g^{ij}) of (5.6a) and (5.6c) are known to the needed accuracy. Will $[Wil\ \ 81]$ has shown that any metric theory of gravitation may be given by a suitable transformation in the so-called “standard form”. For the metric (5.6a) this transformation is given by

$$c\tilde{t} = ct - \frac{1}{2c^3}\frac{\partial \chi}{\partial t},$$

i.e. only by a time-transformation. But it will be shown that there is no necessity for such a transformation as already remarked by Chugreev $[Chu\ 90]$.

5.2 Conservation Laws

When we start instead of (5.6a) from the better approximation for i, j =1,2,3

$$g_{ij} = \left(1 + \frac{2}{c^2}U\right)\delta_{ij} + \frac{1}{c^4}S_{ij}$$

where $S_{ij} = O(1)$ then the energy-momentum tensor (1.21a) can be calculated to the accuracy

$$T(G)^i_j \sim O\left(\frac{1}{c^2}\right) \quad (i, j = 1,2,3)$$
$$\sim O\left(\frac{1}{c}\right) \quad (i=1, 2, 3; j=4), \ (i=4; j=1, 2, 3) \tag{5.16}$$
$$\sim O(1) \quad (i=j=4).$$

Elementary calculations give

$$\begin{aligned} T(G)^i_j &= \frac{1}{\kappa c^4}\left(\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} + \frac{4}{c^2}U\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{4}{c^2}\sum_{k=1}^3\frac{\partial V_k}{\partial x^i}\frac{\partial V_k}{\partial x^j}\right.\\ &\quad \left.-\frac{1}{2c^2}\left(\frac{\partial U}{\partial x^i}\frac{\partial S}{\partial x^j} + \frac{\partial U}{\partial x^j}\frac{\partial S}{\partial x^i}\right) + \delta^i_j\frac{c^4}{16}L_G\right) \quad (i, j=1,2,3)\\ &= \frac{1}{\kappa c^4}\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial ct} \quad (i=1, 2, 3; j=4)\\ &= -\frac{1}{\kappa c^4}\frac{\partial U}{\partial ct}\frac{\partial U}{\partial x^j} \quad (i=4; j=1, 2, 3)\\ &= -\frac{1}{2\kappa c^4}\sum_{k=1}^3\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} \quad (i=j=4) \end{aligned} \tag{5.17a}$$

where

$$\begin{aligned} \frac{c^4}{8}L_G &= -\sum_{k=1}^3\left(\frac{\partial U}{\partial x^k}\right)^2 + \left(\frac{\partial U}{\partial ct}\right)^2 - \frac{4}{c^2}U\sum_{k=1}^3\left(\frac{\partial U}{\partial x^k}\right)^2\\ &\quad + \frac{4}{c^2}\sum_{k=1}^3\left(\frac{\partial V_l}{\partial x^k}\right)^2 + \frac{1}{c^2}\sum_{k=1}^3\frac{\partial U}{\partial x^k}\frac{\partial S}{\partial x^k}. \end{aligned} \tag{5.17b}$$

Hence, the energy-momentum tensor $T(G)^i_j$ of (5.17) is given to the stated accuracy (5.16). It follows that the knowledge of S_{ij} is not necessary.

We will now calculate $T(M)^{ij}$ to the same accuracy as stated by (5.16). It follows from (1.28), (5.4), (5.6) and (5.7) for the symmetric matter tensor

$$\begin{aligned} T(M)^{ij} &= \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right)v^i v^j + pc^2\delta^{ij} \quad (i, j = 1,2,3)\\ &= \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right)cv^i \quad (i = 1,2,3; j = 4)\\ &= \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2\right)c^2. \ (i = j = 4) \end{aligned} \tag{5.18}$$

We obtain from (5.17) by the use of (5.13), (5.2a) and (5.10a)

$$\frac{\partial}{\partial x^k}T(G)_j^k = -\rho\left(1+\frac{\Pi}{c^2}+\frac{4}{c^2}U+2\left|\frac{v}{c}\right|^2+3\frac{p}{\rho}\right)\frac{\partial U}{\partial x^j} + \frac{4}{c^2}\rho\sum_{k=1}^{3}v^k\frac{\partial V_k}{\partial x^j}+\frac{1}{2c^2}\rho\frac{\partial S}{\partial x^j} \tag{5.19a}$$

to accuracy of $O\left(\frac{1}{c^2}\right)$ and

$$\frac{\partial}{\partial x^k}T(G)_4^k = -\frac{1}{c}\rho\frac{\partial U}{\partial t} \tag{5.19b}$$

to accuracy of $O\left(\frac{1}{c}\right)$. It follows from (5.18) and (5.6a)

$$\frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T(M)^{kl} = \rho\left(1+\frac{\Pi}{c^2}+\frac{4}{c^2}U+2\left|\frac{v}{c}\right|^2+3\frac{p}{\rho}\right)\frac{\partial U}{\partial x^j} - \frac{4}{c^2}\rho\sum_{k=1}^{3}v^k\frac{\partial V_k}{\partial x^j}-\frac{1}{2c^2}p\frac{\partial S}{\partial x^j}\ (j=1,2,3) \tag{5.20a}$$

to accuracy $O\left(\frac{1}{c^2}\right)$ and

$$\frac{1}{2}\frac{\partial g_{kl}}{\partial ct}T(M)^{kl} = \frac{1}{c}\rho\frac{\partial U}{\partial t} \tag{5.20b}$$

to accuracy $O\left(\frac{1}{c}\right)$. Hence, we get by comparing (5.19) and (5.20)

$$\frac{\partial}{\partial x^k}T(G)_j^k = -\frac{1}{2}\frac{\partial g_{kl}}{\partial x^j}T(M)^{kl} \quad \text{(j=1-4)} \tag{5.21}$$

to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to $O\left(\frac{1}{c}\right)$ for j=4. The equations of motion (1.29a) to the above noted accuracy are equivalent to the conservation law of energy-momentum (see (1.25a))

$$\frac{\partial}{\partial x^k}\left(T(G)_j^k + T(M)_j^k\right) = 0 \quad \text{(j=1-4).} \tag{5.22}$$

Put

$$P_j = \int\left(T(G)_j^4 + T(M)_j^4\right)d^3x \quad \text{(j=1-4).} \tag{5.23}$$

Hence, P_j is constant to accuracy $O\left(\frac{1}{c}\right)$ for j=1,2,3 and to accuracy $O(1)$ for j=4. It follows from (5.23) with the aid of (5.8a), (5.9a), (5.2a) and (5.12b) by the theorem of Gauß

$$P_j = c\int \rho\left\{v^j\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right)-\frac{4}{c^2}V_j+\frac{1}{2c^2}\frac{\partial^2\chi}{\partial t\partial x^j}\right\}d^3x \quad (5.24a)$$

for j=1,2.3 and

$$P_4 = -c^2\int \rho\left(1+\frac{\Pi}{c^2}+\frac{5}{2c^2}U+\left|\frac{v}{c}\right|^2\right)d^3\,x \quad (5.24b)$$

where the identity

$$\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} = -U\Delta U+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right)$$

is used. Will $[Wil\ 81]$ introduces for j=1,2,3

$$W_j = k\int\frac{\rho\prime\left(v',(x-x\prime)\right)\left(x^j-x^j\prime\right)}{|x-x\prime|^3}d^3x' \quad (5.25a)$$

then (compare also Chandrasekhar $[Cha\ 65]$)

$$\frac{\partial^2\chi}{\partial t\partial x^j} = V_j - W_j. \quad (5.25b)$$

We get from the conservation law for mass

$$\left(\left(\frac{-G}{-\eta}\right)^{1/2}\rho\frac{dx^k}{d\tau}\right)_{/k} = 0 \quad (5.26)$$

by the use of (5.4), (5.6d) and (5.7) the conservation law

$$\frac{\partial\rho^*}{\partial t}+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\rho^*v^k\right) = 0 \quad (5.27a)$$

to $O\left(\frac{1}{c^2}\right)$ where

$$\rho^* = \rho\left(1+\frac{3}{c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2\right). \quad (5.27b)$$

Hence, the conserved mass is given by

$$m = \int\rho^* d^3\,x. \quad (5.28)$$

The conserved energy-momentum follows from (5.24) with (5.27b) and by (5.25)

$$P_j = c\int \rho^* \left[v^j\left(1+\frac{\Pi}{c^2}+\frac{3}{c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right) - \frac{1}{2c^2}\left(7V_j + W_j\right)\right] d^3x \qquad (5.29a)$$

(j=1,2,3)

$$P_4 = -c^2\int \rho^* \left(1+\frac{\Pi}{c^2}-\frac{1}{2c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2\right) d^3\,x. \qquad (5.29b)$$

By the use of the identity (see e.g. [Cha 65])

$$\int \rho U v^j\, d^3x = \int \rho V_j\, d^3x$$

the momentum (5.29a) is rewritten to $O\left(\frac{1}{c}\right)$ in the form

$$P_j = c\int \rho^* \left[v^j\left(1+\frac{\Pi}{c^2}-\frac{1}{2c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right) - \frac{1}{2c^2}W_j\right] d^3x. \qquad (5.29c)$$

The conserved quantities of mass (5.28) and of the energy-momentum (5.29b) and (5.29c) are identical with the corresponding results of Einstein's theory (see [Cha 65] or [Wil 81]).

It is worth mentioning that we have used the energy-momentum tensor in the form (5.1a) with the factor $\left(\frac{-G}{-\eta}\right)^{1/2}$ to get formally the same results as those of general relativity. In general the above factor is omitted which would give the same results in another form of representation.

We will now study the conservation law of angular-momentum (1.53) in uniformly moving reference frames. We get

$$M^{ij} = \int\left(x^i\tilde{T}^{j4} - x^j\tilde{T}^{i4} + A^{ij4}\right) d^3x \qquad (5.30)$$

is conserved for i, j=1,2,3,4. It follows by the use of (5.6) that $A^{ij4} = 0$ to an accuracy of $O\left(\frac{1}{c}\right)$ for i, j=1,2,3 and to an accuracy of $O(1)$ for i=4; j=1,2,3 and i=1,2,3; j=4.

Hence, we obtain to the given accuracy the usual conservation law of angular-momentum, i.e. without spin expression:

$$M^{ij} = \int\left(x^i\tilde{T}^{j4} - x^j\tilde{T}^{i4}\right) d^3x. \qquad (5.31)$$

In particular, for j=4 we get with (5.23)

$$M^{i4} = \int \left(-x^i T_4^4 - ctT_i^4\right) d^3x = -\int x^i\, T_4^4 d^3x - ctP_i. \tag{5.32}$$

If we substitute (5.8a) and (5.17a) into relation (5.32) we get for i=1,2,3 by elementary calculations

$$M^{i4} = c^2 \int x^i \rho^* \left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2} U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right) d^3x - ctP_i. \tag{5.33}$$

Defining the centre of the mass (X^1, X^2, X^3) (see Will [Wil 81]) by

$$X^i = \int x^i \rho^* \left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2} U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right) d^3x \Big/ \int \rho^* \left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2} U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right) d^3x.$$

We get from equation (5.33) by differentiation and the use of (5.29b)

$$\frac{d}{dt} X^i = -c \frac{P_i}{P_4} \quad \text{(i=1,2,3)} \tag{5.34}$$

i.e. the centre of the mass moves uniformly with the velocity $-\frac{c}{P_4}(P_1, P_2, P_3)$.

5.3 Equations of Motion

The equations of motion (1.29a) can be rewritten (see Petry [Pet 91])

$$\frac{\partial}{\partial x^k} T(M)^{jk} = -\Gamma(G)^j_{kl} T(M)^{kl}. \tag{5.35}$$

Elementary calculations give by the use of (5.6), (5.15) and (5.25b) for i, j, k=1,2,3 the Christoffel symbols

$$\begin{aligned}
&\Gamma(G)^4_{44} = -\frac{1}{c^3}\frac{\partial U}{\partial t}, \Gamma(G)^4_{4i} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i},\\
&\Gamma(G)^4_{ij} = \frac{1}{c^3}\left\{\frac{\partial U}{\partial t}\delta_{ij} + 2\left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right)\right\},\\
&\Gamma(G)^i_{44} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i} + \frac{1}{c^4}\left(2\frac{\partial U^2}{\partial x^i} - \frac{\partial \phi}{\partial x^i} - \frac{7}{2}\frac{\partial V_i}{\partial t} - \frac{1}{2}\frac{\partial W_i}{\partial t}\right),\\
&\Gamma(G)^i_{4j} = \frac{1}{c^3}\left\{\frac{\partial U}{\partial t}\delta_{ij} - 2\left(\frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i}\right)\right\},\\
&\Gamma(G)^i_{jk} = \frac{1}{c^2}\left(\frac{\partial U}{\partial x^k}\delta_{ij} + \frac{\partial U}{\partial x^j}\delta_{jk} - \frac{\partial U}{\partial x^i}\delta_{jk}\right).
\end{aligned} \tag{5.36}$$

The equations of motion are satisfied to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to accuracy $O\left(\frac{1}{c}\right)$ for j=4 (see (5.21)). Hence, it follows from formula (5.35) with j=4 that $\Gamma(G)^4_{ij} \approx 0$ (i, j=1,2,3) to the needed accuracy. Therefore, the Christoffel symbols (5.36) with $\Gamma(G)^4_{ij} = 0$ are identical with those of general relativity (see $[Wil\ 81]$ and $[Cha\ 65]$). The equations of motion (5.35) are by the use of (5.18) and (5.36) given to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to accuracy $O\left(\frac{1}{c}\right)$ for j=4. Here, the density ρ^* given by (5.27b) may be introduced instead of the densityρ.

Let $T(M_E)^{ij}$ denote the symmetric matter tensor of the theory of Einstein then we have the relation

$$T(M)^{ij} = \left(\frac{-G}{-\eta}\right)^{1/2} T(M_E)^{ij}. \tag{5.37}$$

The equations of motion of general relativity of Einstein can be written (see e.g. Fock $[Foc\ 60]$)

$$\frac{\partial}{\partial x^k}\left\{\left(-G(E)\right)^{1/2} T(M_E)^{ik}\right\} = -\Gamma(G_E)^i_{kl}\left(-G(E)\right)^{1/2} T(M_E)^{kl} \tag{5.38}$$

where $\Gamma(G_E)^i_{kl}$ are the Christoffel symbols of the theory of Einstein and $G(E)$ is the determinant of the corresponding metric. By virtue of (5.37), $\eta = -1$, $G = G(E)$ to $O\left(\frac{1}{c^2}\right)$ and the agreement of $\Gamma(G)^i_{jk}$ with $\Gamma(G_E)^i_{jk}$ to the needed accuracy the equations of motion (5.35) of gravitation in flat space-time agree with the equations of motion (5.38) of general relativity. Hence, the equations of motion are to post-Newtonian approximation identical with the results of the theory of Einstein.

Summarizing, all the results of flat space-time theory of gravitation and the general theory of relativity of Einstein agree to post-Newtonian approximation.

The results of this chapter on post-Newtonian approximations by the use of the theory of gravitation in flat space-time can be found in the article of Petry $[Pet\ 92]$. Post-Newtonian approximations to higher order (to $2\frac{1}{2}$) are given in the paper of Thümmel $[Th$ü $96]$ by the use of the theory of gravitation in flat space-time.

Chapter 6

Post-Newtonian of Spherical Symmetry

In this chapter spherically symmetric stars with their gravitational fields are studied to post-Newtonian approximation. The equations of motion of the star and the energy-momentum tensor are given. All these results agree with the corresponding results of general relativity to 1-post-Newtonian accuracy whereas to 2-post-Newtonian approximation the results are different from one another. In particular, the theorem of Birkhoff is not valid. Hence, the theory of gravitation in flat space-time and the general theory of relativity are different to this accuracy.

6.1 Post-Newtonian Approximation of Non-Stationary Stars

The equations describing a non-stationary spherically symmetric star depending on the distance from the centre r of the star and the time $\tilde{t}$ are given in chapter III. The field equations are stated in formula (3.12) with $\rho \to \rho\left(1+\frac{\Pi}{c^2}\right)$ where Π denotes the specific internal energy. The equations of motion are stated by (3.13) and the conservation of the whole energy-momentum is given by (3.14). Furthermore, we have an equation of state (3.15).It is worth to mention that the equations of field (3.12) together with the equations of motion (3.13) imply the equations of the whole energy-momentum (3.14). Hence, the relations (3.14) can be omitted. The conservation law of mass (1.29b) has the form

$$\frac{1}{r^2F_{(4)}}\left(\frac{\partial}{\partial r}\left(r^2F_{(4)}\rho u^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2F_{(4)}\rho u^4\right)\right)=0 \tag{6.1}$$

where u^4 is given by (3.6). Hence, we have eight functions f, g, h, F, ρ, p, Π and u^1 depending on r and $\tilde{t}$ and eight independent equations: (3.12) (four equations), (3.13) (two equations), (6.1) (one equation) and (3.15) (one equation).

A suitable combination of the equations of motion (3.13) yields

$$\begin{aligned}&\frac{\partial\Pi}{\partial\tilde{t}}-\frac{pc^2}{\rho^2}\frac{\partial\rho}{\partial\tilde{t}}-\frac{pc^2}{\rho^2}\frac{\partial}{\partial\tilde{t}}\log(fg^2F_{(4)}^2h)^{1/2}\\&+c\frac{u^1}{u^4}\left\{\frac{\partial\Pi}{\partial r}-\frac{pc^2}{\rho^2}\frac{\partial\rho}{\partial r}-\frac{pc^2}{\rho^2}\frac{\partial}{\partial r}\log(fg^2F_{(4)}^2h)^{1/2}\right\}\\&=0\end{aligned} \tag{6.2}$$

Hence, we replace equation (3.13b) by the simpler equation (6.2). In the following we introduce the radial velocity $v(r,\tilde{t})$ instead of u^1 and u^4 given by

$$u^1 = v\frac{d\tilde{t}}{d\tau} = v\left(\frac{h}{1-\frac{h}{f}\left|\frac{v}{c}\right|^2}\right)^{1/2}, \quad u^4 = c\frac{d\tilde{t}}{d\tau} = c\left(\frac{h}{1-\frac{h}{f}\left|\frac{v}{c}\right|^2}\right)^{1/2}. \tag{6.3}$$

The post-Newtonian approximation assumes

$$\begin{gathered}\rho = 0(1),\ v = 0(1),\ \frac{p}{\rho} = 0\left(\frac{1}{c^2}\right),\\ \frac{\Pi}{\rho} = 0\left(\frac{1}{c^2}\right),\ \frac{\partial}{\partial r} = 0(1),\ \frac{\partial}{\partial c\tilde{t}} = 0\left(\frac{1}{c}\right).\end{gathered} \tag{6.4}$$

We make the following ansatz for the post-Newtonian approximation

$$\begin{gathered}f = 1-\frac{2}{c^2}U_1+\frac{1}{c^4}S_1 = 0\left(\frac{1}{c^4}\right),\ g = 1-\frac{2}{c^2}U_2+\frac{1}{c^4}S_2 = 0\left(\frac{1}{c^4}\right)\\ F_{(4)}^2 h = 1+\frac{2}{c^2}U_3+\frac{1}{c^4}S_3 = 0\left(\frac{1}{c^4}\right),\ F = c\tilde{t}+\frac{1}{c^3}S_4 = 0\left(\frac{1}{c^3}\right)\end{gathered} \tag{6.5}$$

Here, the functions U_i (i=1,2,3), S_i(i=1,2,3) are of order $O(1)$ and depend on r and $\tilde{t}$.

The boundary conditions must converge to zero as r goes to infinity. It holds

$$F_{(4)} = 1+\frac{1}{c^4}\frac{\partial S_4}{\partial \tilde{t}} = O\left(\frac{1}{c^4}\right) \tag{6.6a}$$

and

$$h = 1+\frac{2}{c^2}U_3+\frac{1}{c^4}\left(S_3-2\frac{\partial S_4}{\partial \tilde{t}}\right) = O\left(\frac{1}{c^4}\right). \tag{6.6b}$$

The post-Newtonian approximation implies

$$S_1 = S_2 = 0. \tag{6.7}$$

Hence, we get from the field equations (3.12) to $O\left(\frac{1}{c^2}\right)$ that

$$U_1 = U_2 = U_3 = U \tag{6.8}$$

which satisfies the differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = -4\pi k\rho. \tag{6.9a}$$

The third field equation gives to $O\left(\frac{1}{c^4}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(S_3-2U^2)}{\partial r}\right) = 2\frac{\partial^2 U}{\partial \tilde{t}^2} - 8\pi k\rho\left(\Pi + 3\frac{pc^2}{\rho} + 2v^2\right) \tag{6.9b}$$

where $\frac{\partial^2 U}{\partial \tilde{t}^2}$ must be calculated to $O(1)$. The last field equation implies to $O\left(\frac{1}{c^3}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial^2 S_4}{\partial r^2}\right) - \frac{2}{r^2}\frac{\partial S_4}{\partial r} = 16\pi k\rho v.$$

This equation can be integrated by the use of the boundary conditions implying

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S_4}{\partial r}\right) = 16\pi k\int_{-\infty}^{r}\rho v dx. \tag{6.9c}$$

The differential equations (6.9) must be solved by the use of the boundary conditions.

The solutions of (6.9a) and (6.9c) are

$$U = 4\pi k\left\{\frac{1}{r}\int_0^r x^2\rho(x,\tilde{t})dx - \int_\infty^r x\rho(x,\tilde{t})dx\right\}, \tag{6.10a}$$

$$S_4 = \frac{16\pi k}{3}\left\{\frac{1}{r}\int_0^r x^3\rho v dx + \frac{1}{2}r^2\int_\infty^r \rho v dx - \frac{3}{2}\int_\infty^r x^2\rho v dx\right\}. \tag{6.10b}$$

Equation (6.1) gives by the use of (6.3), (6.5) and (6.6a) to $O(1)$

$$\frac{\partial\rho}{\partial\tilde{t}} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = 0.$$

Differentiation of equation (6.9a) gives by the use of this relation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial\tilde{t}}\right)\right) = -4\pi k\frac{\partial\rho}{\partial\tilde{t}} = 4\pi k\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v).$$

Elementary integration yields to $O(1)$

$$\frac{\partial U}{\partial\tilde{t}} = 4\pi k\int_\infty^r \rho v dx. \tag{6.11}$$

Equation (6.9c) gives by differentiation and the use of (6.11)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\partial S_4}{\partial\tilde{t}}\right)\right) = 16\pi k\frac{\partial}{\partial\tilde{t}}\int_\infty^r \rho v dx = 4\frac{\partial^2 U}{\partial\tilde{t}^2}.$$

Therefore, equation (6.9b) can be rewritten in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(S_3-2U^2-\frac{1}{2}\frac{\partial S_4}{\partial \tilde{t}}\right)\right)=-8\pi k\rho\left(\Pi+3\frac{pc^2}{\rho}+2v^2\right) \quad (6.12)$$

Let us now introduce the potentials in analogy to (5.14)

$$\begin{aligned}\phi_1&=4\pi k\left(\frac{1}{r}\int_0^r x^2\rho v^2 dx-\int_\infty^r x\rho v^2 dx\right),\\ \phi_3&=4\pi k\left(\frac{1}{r}\int_0^r x^2\rho\Pi dx-\int_\infty^r x\rho\Pi dx\right),\\ \phi_4&=4\pi k\left(\frac{1}{r}\int_0^r x^2 pc^2 dx-\int_\infty^r xpc^2 dx\right).\end{aligned} \quad (6.13)$$

The differential equation (6.12) has the solution

$$S_3=2U^2+\frac{1}{2}\frac{\partial S_4}{\partial \tilde{t}}+2(2\phi_1+\phi_3+3\phi_4). \quad (6.10c)$$

The relations (6.10) give together with (6.5), (6.6), (6.7) and (6.8) the post-Newtonian approximation.

The energy-momentum tensor of matter (3.7) can now be given to accuracy

$$\begin{aligned}T(M)^i_j&=0\left(\frac{1}{c^2}\right),(i=j=1-4)\\ &=0\left(\frac{1}{c^3}\right),(i=1;\ j=4)\\ &=0\left(\frac{1}{c}\right),(i=4;\ j=1)\end{aligned} \quad (6.14a)$$

and the corresponding tensor of the gravitational field

$$\begin{aligned}T(G)^i_j&=0\left(\frac{1}{c^2}\right),(i=j=1,2,3,4)\\ &=0\left(\frac{1}{c^3}\right),(i=1;\ j=4),(i=4;\ j=1)\end{aligned} \quad (6.14b)$$

The expressions of these tensors to post-Newtonian approximation are omitted but they can be found in the article [$Pet\ 94a$].

We will now give the equations of motion to post-Newtonian accuracy of $O\left(\frac{1}{c^2}\right)$. We use the differential equations (3.13) with matter the tensor (3.7), the differential equation (6.2) and the conservation law of mass (6.1). The post-Newtonian approximations (6.5) with (6.6), (6.7) and (6.8) are used. Furthermore, the representations (6.10) are introduced. After longer calculations the following post-Newtonian approximations to $O\left(\frac{1}{c^2}\right)$ are received:

$$\begin{aligned}\frac{\partial\rho}{\partial\tilde{t}} &= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v)\\ &+\frac{1}{c^2}\left[v\frac{\partial pc^2}{\partial r}+4\pi k\left(2\frac{v}{r^2}\int_0^r x^2\rho dx+\int_\infty^r v\rho dx\right)\right]\end{aligned} \tag{6.15a}$$

$$\begin{aligned}\frac{\partial\Pi}{\partial\tilde{t}} &= -v\frac{\partial\Pi}{\partial r}\\ &+\frac{pc^2}{\rho^2}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)+\frac{1}{c^2}\left(v\frac{1}{\rho}\frac{\partial pc^2}{\partial r}+4\pi k\left(4\frac{v}{r^2}\int_0^r x^2\rho dx+3\int_r^\infty v\rho dx\right)\right)\right]\end{aligned} \tag{6.15b}$$

$$\begin{aligned}\frac{\partial v}{\partial\tilde{t}} &= -v\frac{\partial v}{\partial r}-\frac{1}{\rho}\frac{\partial pc^2}{\partial r}\left[1-\frac{1}{c^2}\left(\Pi+\frac{pc^2}{\rho}+4U+2v^2\right)\right]\\ &+\frac{\partial U}{\partial r}\left(1+\frac{2}{c^2}\left(\frac{pc^2}{\rho}-2U\right)\right)\\ &-\frac{4\pi k}{c^2}r\int_r^\infty\frac{1}{x}\rho\left(2v^2+\frac{4\pi k}{x}\int_0^x y^2\rho(y,\tilde{t})dy\right)dx\\ &-\frac{4\pi k}{c^2}\frac{1}{r^2}\int_0^r x^2\rho\left(v^2+\Pi+\frac{4\pi k}{x}\int_0^x y^2\rho(y,\tilde{t})dy\right)dx\\ &+\frac{v}{c^2}\frac{1}{r^2}\frac{\partial(r^2v)}{\partial r}\left(\frac{pc^2}{\rho}+\frac{pc^2}{\rho}\frac{\partial}{\partial\Pi}\left(\frac{pc^2}{\rho}\right)+\rho\frac{\partial}{\partial\rho}\left(\frac{pc^2}{\rho}\right)\right)\\ &+\frac{12\pi k}{c^2}v\left(\frac{v}{r^2}\int_0^r x^2\rho dx+\int_r^\infty\rho v dx\right).\end{aligned} \tag{6.15c}$$

In the equation (6.15c) we have to eliminate U by relation (6.10a). Then, the equations (6.15) are three integro-differential equations to post-Newtonian approximation $O\left(\frac{1}{c^2}\right)$ for the three unknown functions v,ρ and Π. Let us assume an equation of state

$$\frac{p}{\rho} = \Pi\, B(\Pi,\rho) \tag{6.16}$$

with a suitable function B. The boundary $R(\tilde{t})$ of the star follows from (6.15b) with $r = R(\tilde{t})$:

$$\frac{d}{d\tilde{t}}\left(\Pi(R(\tilde{t}),\tilde{t})\right) = \left(\frac{\partial\Pi(r,\tilde{t})}{\partial\tilde{t}} + v\frac{\partial\Pi(r,\tilde{t})}{\partial r}\right)\left(r = R(\tilde{t})\right) \sim p\left(R(\tilde{t})\right) = 0.$$

Therefore, we have that

$$\Pi(R(\tilde{t}),\tilde{t}) = \Pi(R(\tilde{t}_0),\tilde{t}_0) = 0 \tag{6.17}$$

if for a fixed time $\tilde{t}_0$ the relation $\Pi(R(\tilde{t}_0),\tilde{t}_0) = 0$ holds. Then, relation (6.17) defines the boundary of the non-stationary star to post-Newtonian accuracy. The equation (6.17) is independent of the equation of state (6.16) but (6.16) is in agreement with (6.17).

The detailed longer derivations of the equations (6.15) are given in $[Pet\ 94a]$.

We will now study the potentials in the exterior of the star, i.e. $r > R(\tilde{t})$. It follows from relation (6.5) with (6.6), (6.7), (6.8) and (6.10)

$$f \approx g \approx 1 - \frac{2}{c^2}U \tag{6.18a}$$

$$h \approx 1 + \frac{2}{c^2}U + \frac{1}{c^4}\left(2U^2 - \frac{3}{2}\frac{\partial S_4}{\partial\tilde{t}} + 2(2\phi_1 + \phi_3 + 3\phi_4)\right) \tag{6.18b}$$

with

$$U = \frac{4\pi k}{r}\int_0^\infty x^2\rho dx$$

$$\frac{\partial S_4}{\partial\tilde{t}} = \frac{16\pi k}{3}\frac{1}{r}\int_0^\infty x^3\frac{\partial(\rho v)}{\partial\tilde{t}}dx$$

$$2\phi_1 + \phi_3 + 3\phi_4 = \frac{4\pi k}{r}\int_0^\infty x^2\rho\left(2v^2 + \Pi + 3\frac{pc^2}{\rho}\right)dx.$$

It holds (see $[Pet\ 94a]$) that the gravitational mass to $O\left(\frac{1}{c^2}\right)$ is

$$M_g = 4\pi\int_0^\infty x^2\rho\left\{1 + \frac{1}{c^2}\left(\Pi + \frac{1}{2}U + v^2\right)\right\}dx \tag{6.19a}$$

Hence, relation (6.18a) gives to $O\left(\frac{1}{c^2}\right)$

$$f \approx g \approx 1 - 2\frac{kM_g}{c^2 r} \tag{6.19b}$$

Relation (6.18b) can be rewritten by the use of (6.19a) to $O\left(\frac{1}{c^4}\right)$

$$h \approx 1 + 2\frac{kM_g}{c^2 r} + \frac{1}{c^4}\left\{2\left(\frac{kM_g}{c^2 r}\right)^2 - \frac{8\pi k}{r}\int_0^\infty \left(x^3 \frac{\partial(\rho v)}{\partial \tilde{t}} - x^2 \rho\left(v^2 + 3\frac{pc^2}{\rho} - \frac{1}{2}U\right)\right)dx\right\}.$$

In the article $[Pet\ 94a]$ it is shown that the last expression vanishes. Therefore, we get to $O\left(\frac{1}{c^4}\right)$

$$h \approx 1 + 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2. \tag{6.19c}$$

The relations (6.19) show that in the exterior of the star the theorem of Birkhoff holds to post-Newtonian accuracy.

6.2 2-Post-Newtonian Approximation of a Non-Stationary Star

We will in this sub-chapter only give some results of 2-post-Newtonian approximation. The study is given in $[Pet\ 94b]$ where the results are derived. We make the ansatz

$$\begin{aligned} &f = 1 - \frac{2}{c^2}U + \frac{1}{c^4}S_1, g = 1 - \frac{2}{c^2}U + \frac{1}{c^4}S_2, \\ &F_{(4)}^2 h = 1 + \frac{2}{c^2}U + \frac{1}{c^4}S_3 + \frac{1}{c^6}S_5, F = c\tilde{t} + \frac{1}{c^3}S_4 + \frac{1}{c^5}S_6 \end{aligned} \tag{6.20}$$

where S_1, S_2, S_5 and S_6 are of order $O(1)$ and U, S_3 and S_4 are already given in chapter 6.1.

For 2-post-Newtonian the time-derivatives must be considered to higher approximations, i.e. let $y(r, \tilde{t})$ be any function then the following approximation is used

$$\frac{\partial y}{\partial \tilde{t}} = \left(\frac{\partial y}{\partial \tilde{t}}\right)_0 + \frac{1}{c^2}\left(\frac{\partial y}{\partial \tilde{t}}\right)_2 + \frac{1}{c^4}\left(\frac{\partial y}{\partial \tilde{t}}\right)_4 \tag{6.21}$$

where $\left(\frac{\partial y}{\partial \tilde{t}}\right)_0$ is the Newtonian approximation, $\left(\frac{\partial y}{\partial \tilde{t}}\right)_2 = O\ (1)$ and $\left(\frac{\partial y}{\partial \tilde{t}}\right)_4 = O\ (1)$ are the 1-post-Newtonian and the 2-post-Newtonian approximations. We get from (6.20) up to 2-post-Newtonianaccuracy

$$F_{(4)} = \frac{\partial F}{\partial c\tilde{t}} = 1 + \frac{1}{c^4}\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0 + \frac{1}{c^6}\left\{\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_2 + \left(\frac{\partial S_6}{\partial \tilde{t}}\right)_0\right\} \tag{6.22a}$$

$$h = 1 + \frac{2}{c^2}U = \frac{1}{c^4}\left\{S_3 - 2\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0\right\} + \frac{1}{c^6}\left\{S_5 - 4U\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0 - 2\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_2 - 2\left(\frac{\partial S_6}{\partial \tilde{t}}\right)_0\right\}. \tag{6.22b}$$

In analogy to chapter 6.1 the expressions (6.20) and (6.22) are substituted into the differential equations (3.12). We get by elementary longer calculations differential equations for the 2-post-Newtonian approximations S_1, S_2, S_5 and S_6 whereas the functions U, S_4 and S_3 are given by (6.10). The solutions of these equations are given as functions of ρ, p, Π and v. It is worth to mention that S_6 implies divergent integrals by the standard 2-post Newtonian approximation. Hence, it is necessary to use retarded functions. Therefore, the expression of the energy tensor contains retardations implying gravitational waves of the order $O\left(\frac{1}{c^5}\right)$. This is a well-known fact of higher order post-Newtonian approximations also by the use of general relativity theory. This may be the reason why higher order post-Newtonian approximations are not possible implying divergent integrals. Furthermore, the expressions for S_6, $\frac{\partial S_6}{\partial r}$ and $\frac{\partial S_6}{\partial \tilde{t}}$ are not of order $O(1)$. Therefore, they do not fulfil the condition on 2-post-Newtonian approximation. The whole energy-momentum tensors of matter and of gravitational field can be given. The equations of motion and the conservation of mass are also stated where the gravitational mass M_g can be given to accuracy $O\left(\frac{1}{c^6}\right)$. We will now state the solution of the non-stationary star in the exterior, i.e. $r > R(\tilde{t})$.

It follows

$$\begin{aligned}
f &= 1 - 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 \\
&\quad + \frac{1}{c^4}\left\{\frac{16}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dy dx - \frac{8}{15}\frac{4\pi k}{r^3}\int_0^\infty x^4\rho v^2 dx\right\}, \\
g &= 1 - 2\frac{kM_g}{c^2 r} + 3\left(\frac{kM_g}{c^2 r}\right)^2 \\
&\quad - \frac{1}{c^4}\left\{\frac{8}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dy dx - \frac{4}{15}\frac{4\pi k}{r^3}\int_0^\infty x^4\rho v^2\, dx\right\}, \\
h &= 1 + 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 + 2\left(\frac{kM_g}{c^2 r}\right)^3
\end{aligned} \tag{6.23}$$

$$+\frac{1}{c^6}\left\{-\frac{8}{15}\frac{kM_g}{r}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx+\frac{4}{15}\frac{kM_g}{r}\frac{4\pi k}{r^3}\int_0^\infty x^4\,\rho v^2dx\right\}$$

$$+\frac{1}{c^6}\left\{\frac{8}{9}\frac{(4\pi k)^2}{r^4}\left(\int_0^\infty x^3\,\rho vdx\right)^2-\frac{8}{9}\frac{kM_g}{r}\frac{4\pi k}{r}\left(\frac{\partial}{\partial\tilde{t}}\int_0^\infty x^3\,\rho vdx\right)_0\right\}$$

$$-\frac{1}{c^6}\frac{8}{15}\frac{4\pi k}{r}\left(\frac{\partial}{\partial\tilde{t}}\int_0^\infty x^3\rho\left(4\frac{pc^2}{\rho}v+v^3+x\frac{pc^2}{\rho}\frac{\partial v}{\partial x}\right)dx\right)_0$$

$$+\frac{1}{c^6}\left\{\frac{(4\pi k)^2}{r}\left(\frac{\partial}{\partial\tilde{t}}\left[\frac{32}{9}\int_0^\infty x^4\rho\int_\infty^x \rho vdydx+\frac{176}{15}\int_0^\infty x^2\rho v\int_0^x y^2\rho dydx\right]\right)_0\right\}$$

$$+\frac{1}{c^6}\frac{(4\pi k)^2}{r}\left(\frac{\partial}{\partial\tilde{t}}\left(\frac{64}{9}\int_0^\infty x\rho\int_0^x y^3\,\rho vdydx\right)\right)_0.$$

The derivations of all the mentioned results of chapter 6.2 are longer calculations and they are not trivial. Therefore, only the exterior potentials (6.23) of the star are stated. It immediately follows from (6.23) that Birkhoff's theorem is not valid by the use of 2-post-Newtonian approximation. This is in contrast to the theory of Einstein. Hence, flat space-time theory of gravitation and the general relativity theory of gravitation give different results to higher order approximations.

The equations (6.23) give for a static spherically symmetric star up to 2-post-Newtonian approximation

$$\begin{aligned}
f&=1-2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+\frac{1}{c^4}\frac{16}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx,\\
g&=1-2\frac{kM_g}{c^2r}+3\left(\frac{kM_g}{c^2r}\right)^2-\frac{1}{c^4}\frac{8}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx,\\
h&=1+2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+2\left(\frac{kM_g}{c^2r}\right)^3\\
&\quad-\frac{1}{c^6}\frac{8}{15}\frac{kM_g}{r}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx
\end{aligned}\tag{6.24}$$

It follows by comparing the two solutions (6.14) and (2.39) that the constant A of (2.39) is of order $O(c^2)$ with

$$A=\frac{8}{15}\frac{(4\pi c)^2}{kM_g^3}\int_0^\infty x^3\rho\int_0^x y^2\,\rho dydx\,.\tag{6.25}$$

It is worth mentioning that the factors of the expressions $\left(\frac{K}{r}\right)^3$ in the formulae (2.39a) and (2.39b) are of order $O(1)$ and the factor of the expression $\left(\frac{K}{r}\right)^4$ in (2.39c) is of orderof $O(1), too$.But by virtue of (6.25) in the formulae (6.24) these factors are too great and do not satisfy 2-post-Newtonian approximation. Therefore, the exterior solution of a static spherically symmetric star is approximately given by

$$\begin{aligned} f &= 1-2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+2A\left(\frac{kM_g}{c^2r}\right)^3, \\ g &= 1-2\frac{kM_g}{c^2r}+3\left(\frac{kM_g}{c^2r}\right)^2-A\left(\frac{kM_g}{c^2r}\right)^3, \\ h &= 1+2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+2\left(\frac{kM_g}{c^2r}\right)^3-A\left(\frac{kM_g}{c^2r}\right)^4 \end{aligned} \qquad (6.26)$$

where $A = (c^2)$ is stated by formula (6.25). Estimates of A fulfil the condition

$$A \gg 1$$

which is in agreement of (2.39) with (6.26).

All these results with detailed calculations are given in the article $[Pet\ 94b]$.

6.3 Non-Stationary Star and the Trajectory of a Circulating Body

In this sub-chapter a simple model of a non-stationary star is given. The solution contains small time-dependent exterior gravitational effects. The perturbed equations of motion of a test body moving around the non-stationary star are given. The test body moves away from the centre of the star during the epoch of collapsing star and it moves towards the centre during the epoch of expanding star.

The equations of a non-stationary spherically symmetric homogeneous star to Newtonian accuracy as special case of chapter 6.1 (see $[Pet\ 94a]$) are:

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial r} - \frac{1}{\rho}\frac{\partial pc^2}{\partial r} - \frac{4\pi k}{r^2}\int_0^r x^2\,\rho dx, \qquad (6.28a)$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) \qquad (6.28b)$$

$$\frac{\partial \Pi}{\partial t} = -v\frac{\partial \Pi}{\partial r} - \frac{pc^2}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v). \tag{6.28c}$$

The equation of state for a non-relativistic degenerate Fermi gas is

$$\frac{pc^2}{\rho} = \frac{2}{3}\Pi. \tag{6.29}$$

Furthermore, it is assumed that the star is homogeneous, i.e.

$$\rho = \tilde{\rho}(t). \tag{6.30}$$

We use the ansatz

$$v(r,t) = \frac{r}{R_0}\tilde{v}(t), \tag{6.31a}$$

$$\Pi(r,t) = \frac{R^2(t)-r^2}{R_0^2}\tilde{\Pi}(t) \tag{6.31b}$$

where $R(t)$ denotes the radius of the starand R_0 is a fixed arbitrary constant. The gravitational mass to Newtonian accuracy is

$$M_g = 4\pi\int_0^{R(t)} x^2\,\tilde{\rho}dx = \frac{4\pi}{3}\tilde{\rho}R^3. \tag{6.32}$$

It follows by the use of (6.29) to (6.32)

$$\tilde{v} = \frac{R_0}{R(t)}\frac{dR(t)}{dt}, \tag{6.33}$$

$$\tilde{\Pi} = \beta c^2\left(\frac{R_0}{R(t)}\right)^4 \tag{6.34}$$

where β is a constant of integration. Furthermore, the following differential equation $[Pet\ 95a]$ is received

$$\frac{d^2R(t)}{dt^2} = \frac{4}{3}\beta c^2 R_0^2\frac{1}{R(t)^3} - \frac{kM_g}{R(t)^2}. \tag{6.35}$$

This differential equation can be integrated yielding

$$\left(\frac{dR}{dt}\right)^2 = C - \frac{4}{3}\beta c^2\left(\frac{R_0}{R}\right)^2 + 2\frac{kM_g}{R} \tag{6.36}$$

where C is an constant of integration. Knowing a solution $R(t)$ of (6.36) the relations for $\tilde{v}$, $\tilde{\Pi}$ and $\tilde{\rho}$ are obtained by (6.33), (6.34) and (6.32).

There are two different kinds of solutions:

(1) $C \geq 0$: The radius $R(t)$ contracts to a positive minimum and then it expands for all times.

(2) $C < 0$: The radius $R(t)$ of the star oscillates between a mimimum radius R_1 and a maximum radius R_2. They are given by

$$R_1 = \left(kM_g - \left(\left(kM_g \right)^2 - \frac{4}{3} \beta c^2 R_0^2 |C| \right)^{1/2} \right) \Big/ |C|$$
$$R_1 = \left(kM_g + \left(\left(kM_g \right)^2 - \frac{4}{3} \beta c^2 R_0^2 |C| \right)^{1/2} \right) \Big/ |C| \tag{6.37}$$

The relations (6.37) give

$$\frac{1}{2}(R_1 + R_2) = \frac{kM_g}{|C|}. \tag{6.38}$$

Equation (6.38) fixes $|C|$ by the mass and the maximum and minimum radius of the star.

The approximate solution of (6.36) has the form

$$R(t) \approx \frac{1}{2}(R_1 + R_2) - \frac{1}{2}(R_2 - R_1) cos\left(sqrt\left(\frac{kM_g}{((R_1 + R_2)/2)^3} \right) t \right). \tag{6.39}$$

Hence, the solution (6.39) describes to Newtonian accuracy a non-singular spherically symmetric, homogeneous, pulsating star.

The period of the oscillation is

$$t_p = 2\pi \sqrt{R_m^3 / kM_g} \tag{6.40a}$$

where

$$R_m = \frac{1}{2}(R_1 + R_2) \tag{6.40b}$$

is the mean radius of the oscillating object. Formula gives for the Sun with

$$k \approx 6.67 \cdot 10^{-8}[cm/(gsec^2)],\ M_{\odot} \approx 2 \cdot 10^{33}[g],\ R_m \approx 6.96 \cdot 10^{10}[cm]$$

the period of oscillation

$$t_p \approx 9.98 \cdot 10^3 [sec] \approx 166[min]. \tag{6.41}$$

This result is in good agreement with the experimentally measured value of 160 $[min]$.

The special case $R_1 = R_2$ implies by the use of (6.7) the relation

$$\left(kM_g\right)^2 = \frac{4}{3}\beta c^2 R_0^2 |C|.$$

Then, we get with $\tilde{R} = (R_1 + R_2)/2$ by the use of (6.38)

$$|C| = \frac{kM_g}{\tilde{R}}.$$

Hence, the acceleration (6.35) and the velocity (6.36) at $R(t) = \tilde{R}$ are zero, i.e., we have a stationary star with radius $\tilde{R}$. This result also follows by the use of (6.39). The last two relations give

$$\frac{kM_g}{\tilde{R}} = \frac{4}{3}\beta c^2 \left(\frac{R_0}{\tilde{R}}\right)^2.$$

Relation (6.34) implies for $R(t) = \tilde{R}$

$$\tilde{\Pi} = \frac{3}{4}\frac{kM_g}{\tilde{R}}\left(\frac{R_0}{\tilde{R}}\right)^2.$$

At the centre of the star we get

$$\Pi = \left(\frac{\tilde{R}}{R_0}\right)^2 \tilde{\Pi} = \frac{3}{4}\frac{kM_g}{\tilde{R}}.$$

Hence, we have at the centre of the star by the use of (6.29)

$$\frac{p}{\rho} = \frac{1}{2}\frac{kM_g}{c^2\tilde{R}}.$$

Therefore, we receive a non-singular, spherically symmetric, stationary star where the pressure is given by the above relation.

We will now give the exterior gravitational field of a spherically, non-stationary star to 2-post-Newtonian approximation. The potentials in spherical coordinates are given by (3.4b). We get by (6.23) up to $O\left(\frac{1}{r}\right)$

$$\begin{aligned} f &\approx g \simeq 1-2\frac{kM_g}{c^2 r}, \\ h &\approx 1+2\frac{kM_g}{c^2 r}+\frac{1}{c^6}\frac{4\pi k}{r}\frac{\partial \tilde{h}}{\partial t}, \\ \tilde{h} &= -\frac{8}{15}\int_0^R x^3\rho\left(4\frac{pc^2}{\rho}v+v^3+x\frac{pc^2}{\rho}\frac{\partial v}{\partial x}\right)dx \\ &+4\pi k\frac{64}{9}\int_0^R x\rho\int_0^x y^3\rho v dy dx \\ &+4\pi k\left(\frac{32}{9}\int_0^R x^4\rho\int_R^x \rho v dy dx+\frac{176}{15}\int_0^R x^2\rho v\int_0^x y^2\rho dy dx\right) \end{aligned} \tag{6.41a}$$

Elementary calculations yield by the use of (6.30), (6.31), (6.33), (6.34), (6.32) and (6.36) the approximate value

$$\tilde{h} = \frac{8}{35}\frac{M_g}{4\pi}\left(24\frac{kM_g}{R(t)}-C\right)\frac{dR(t)}{dt}. \tag{6.41b}$$

We will now give the motion of a test particle in this gravitational field. The differential equations (2.53) imply by the use of (6.41) for the perturbed orbit $r_0+\Delta r$ around a circle with radius r_0 after some longer calculations (see $[Pet\ 95a]$) the equations

$$\frac{d^2\Delta r}{dt^2} = -\frac{kM_g}{r_0^3}\Delta r - \frac{1}{c^4}\frac{2\pi k}{r_0}\frac{\partial \tilde{h}}{\partial t}. \tag{6.42}$$

This differential equation can be solved by standard methods. We get by suitable initial conditions and elementary longer calculations the perturbed radius

$$\Delta r(t) = -\frac{20}{7}\left(\frac{kM_g}{c^2 r_0}\right)^2 (R(t)-R_1) \tag{6.43a}$$

and the perturbed radial velocity

$$\frac{d}{dt}\Delta r = -\frac{20}{7}\left(\frac{kM_g}{c^2 r_0}\right)^2 \frac{R_2-R_1}{R_2+R_1}\left(\frac{2kM_g}{R_1+R_2}\right)^{1/2} sin\left(\frac{2\sqrt{2kM_g}}{\sqrt[3]{R_1+R_2}}t\right). \tag{6.43b}$$

The derivation of the perturbed solution is given in $[Pet\ 95a]$ where a factor in the denominator is missing.

Hence, the deviations of the orbit and its velocity from a circle are very small. But this result although very small differs from the corresponding results of general relativity where by the theorem of Birkhoff no change of the orbit arises.

All these results are contained in the articles [Pet 95a] and [Pet 10a].

6.4 Gravitational Radiation from a Binary System

In this sub-chapter 1-post-Newtonian approximations are used to derive the gravitational radiation of a system of objects at large distances from one another. A more explicit formula is given for a binary system. It agrees with the result of general relativity.

We use the 1-post-Newtonian approximation of the potentials (5.8) and the tensor of matter

$$\begin{aligned}
T(M)^i_j &= \rho\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right)v^i v^j \\
&\quad + pc^2\left(1+\frac{2}{c^2}U\right)\delta^i_j-\frac{4}{c^2}\rho V_j v^i, (i; j=1,2,3) \\
&= \rho c v^j\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right)-\frac{4}{c}\rho V_j, (i=4; j=1,2,3) \\
&= -\rho c v^i\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right), (i=1,2,3; j=4) \\
&= -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\left(\frac{v}{c}\right)^2\right). (i=j=4)
\end{aligned} \tag{6.44}$$

Here, the potentials U and V_i are stated by (5.2b) and (5.11). Subsequently, we use the tensors (1.32), the field equations (1.34) and the tensors of the gravitational energy (1.35) and of matter (1.37). It follows from (1.34) by multiplication with f^{ki}

$$\left(f^{kl}f^{ij}_{/l}\right)_{/k} = f^{kl}f_{mn}f^{im}_{/k}f^{jn}_{/l} + 4\kappa f^{ik}T^j_k \tag{6.45}$$

Put

$$f^{ij} = \eta^{ij} + \phi^{ij} \tag{6.46}$$

then we get

$$\eta^{kl}\phi^{ij}_{/kl} = \tau^{ij} \tag{6.47a}$$

with

$$\tau^{ij} = -\left(\phi^{kl}\phi^{ij}_{/l}\right)_{/k} + f^{kl}f_{mn}\phi^{im}_{/k}\phi^{jn}_{/l} + 4\kappa f^{ik}T^{j}_{k}. \tag{6.47b}$$

In the following we use the pseudo-Euclidean geometry (1.1) and (1.5). Then, the differential equation (6.47) has the familiar form of a wave equation. The solution for out-going waves is

$$\phi^{ij} = -\frac{1}{4\pi}\int \tau^{ij}\left(x', t - \frac{|x-x'|}{c}\right)/|x-x'|\,dx' \tag{6.48}$$

where the integration is taken over the whole space R^3.

Longer calculations are given in the article of Petry [Pet 93a]. They follow along the lines of the papers [Eps 75], [Wag 76] and [Wil 77] in studying gravitational radiation by the use of general relativity. The resulting radiation energy E per unit time is given to $O\left(\frac{1}{c^5}\right)$ by

$$\frac{dE}{dt} = -\frac{k}{15c^5}\left\{3\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ij}}{dct^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ij}}{\partial ct^2}\right) - \left(\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ii}}{\partial ct^2}\right)\right)^2\right\} \tag{6.49}$$

where, I^{ij} are the quadrupole moments.

It holds for several point masses $m_A$with velocities $v_a = (v_A^1, v_A^2, v_A^3)$:

$$\left(\frac{\partial^2 I^{ij}}{\partial ct^2}\right) = \sum_A m_A\left(2v_A^i v_A^j + \frac{dv_A^i}{dt}x_A^j + x_A^i\frac{dv_A^j}{dt}\right). \tag{6.50}$$

The application of (6.49) and (6.50) to a binary system gives the gravitational radiation

$$\frac{dE}{dt} = -\frac{8}{15}\frac{k^3\mu^2 m^2}{c^5 r^4}\left(12|v|^2 - 11\left(\frac{dr}{dt}\right)^2\right) \tag{6.51}$$

with the following abbreviations for the two objects A and$B$:

$$m = m_A + m_B,\ \mu = \frac{m_A m_B}{m},\ r = |x_A - x_B|, v = v_A - v_B. \tag{6.52}$$

This result is identical with that of the general relativity theory of Einstein to this accuracy (see [Wag 76] and [Wil 77]. Therefore, the results of both theories agree in the magnitude of the gravitational energy emitted by the binary pulsar system PSR 1913+16 (see Taylor [Tay 79]).

All these results can be found for flat space-time theory of gravitation in [Pet 93a] and for the theory of general relativity in the papers [Eps 75], [Wag 76] and [Wil 77].

Chapter 7

The Universe

In this chapter homogeneous, isotropic cosmological models are studied. The differential equations which describe these models together with the solutions are stated. Flat space-time theory of gravitation implies non-singular cosmological solutions, i.e. a big bang does not exist. This chapter follows along the lines of the articles $|Pet\ 90a|$ and $[Pet\ 90b]$.

7.1 Homogeneous Isotropic Cosmological Models with Cosmological Constant

In this chapter homogeneous isotropic cosmological models are studied. The field equations are given and the solutions are derived. There is no big bang.

We tart from the flat space-time theory of gravitation stated in chapter I. We use the flat space-time metric (1.1) with the pseudo-Euclidean geometry (1.5). The tensors of matter $T(M)_j^i$, of radiation $T(R)_j^i$ and of the cosmological constant $T(\Lambda)_j^i$ are given by (1.28) with

$$p = 0 \quad \text{(dust)} \tag{7.1a}$$

with

$$p = \frac{1}{3}\rho \quad \text{(radiation)} \tag{7.1b}$$

and by (1.21b) with (1.10). The energy-momentum tensor of gravitation is stated by relation (1.21a).

In the following, it is assumed that the universe is homogeneous and isotropic and matter is described in the rest frame i.e.

$$u^i = 0 \quad \text{(i=1,2,3).} \tag{7.2}$$

Then, the potentials are described by two time-dependent functions $a(t)$ and $h(t)$ with

$$\begin{aligned} g_{ij} &= a^2(t), (i = j = 1,2,3) \\ &= -1/h(t), (i = j = 4) \\ &= 0. (i \neq j) \end{aligned} \tag{7.3}$$

Then, the differential equations (1.24) with (1.23) yield

$$\left(a^3\sqrt{h}\frac{a'}{a}\right)' = 2\kappa c^4\left(\frac{1}{2}\rho_m + \frac{1}{3}\rho_r + \frac{\Lambda}{2\kappa c^2}\frac{a^3}{\sqrt{h}}\right) \tag{7.4a}$$

$$\left(a^3\sqrt{h}\frac{h'}{h}\right)' = 4\kappa c^4\left(\frac{1}{2}\rho_m + \rho_r - \frac{\Lambda}{2\kappa c^2}\frac{a^3}{\sqrt{h}} + \frac{1}{8\kappa c^2}L_G\right) \tag{7.4b}$$

where

$$L_G = \frac{1}{c^2}a^3\sqrt{h}\left(12\left(\frac{a'}{a}\right)^2 + \left(\frac{h'}{h}\right)^2 - \frac{1}{2}\left(-6\frac{a'}{a} + \frac{h'}{h}\right)^2\right) \tag{7.4c}$$

Here, ρ_m and ρ_r are the densities of matter and radiation, where the relations (7.1) are used and the prime denotes the t-derivative. It follows from (1.12) by the use of relation (7.2)

$$(u^i) = \left(\frac{dx^i}{d\tau}\right) = \left(0,0,0,c\sqrt{h}\right). \tag{7.5}$$

The proper-time is given by (1.8) implying

$$(cd\tau)^2 = -a^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \frac{1}{h}(dct)^2. \tag{7.6}$$

The tensor of matter plus radiation has the form

$$\begin{aligned} T(M)^i_j + T(R)^i_j &= \frac{1}{3}\rho_r c^2 (i = j = 1,2,3) \\ &= -(\rho_m + \rho_r)c^2 (i = j = 4) \\ &= 0 (i \neq j) \end{aligned} \tag{7.7a}$$

and the tensor of gravitation is

$$\begin{aligned} T(G)^i_j &= \frac{1}{16\kappa}L_G \quad (i=j=1, 2, 3) \\ &= -\frac{1}{16\kappa}L_G \quad (i=j=4) \\ &= 0. \quad (i\neq j) \end{aligned} \tag{7.7b}$$

The conservation law of the whole energy-momentum (1.25a) yields with i=4 by the use of (1.23c)

$$(\rho_m + \rho_r)c^2 + \frac{1}{16\kappa}L_G + \frac{\Lambda}{2\kappa}\frac{a^3}{\sqrt{h}} = \lambda c^2 \tag{7.8}$$

where λ is a constant of integration. It follows from the equations of motion (1.26)

$$\frac{d}{dt}(\rho_m + \rho_r) = -3\frac{a'}{a}p_r - \frac{1}{2}\frac{h\prime}{h}(\rho_m + \rho_r). \tag{7.9}$$

Let us assume that matter and radiation are decoupled then (7.9) gives with the initial conditions at present time $t_0 = 0$

$$a(t_0) = h(t_0) = 1 \tag{7.10}$$

the solutions

$$\rho_m = \rho_{m0}/\sqrt{h}, \quad \rho_r = \rho_{r0}/\left(a\sqrt{h}\right) \tag{7.11}$$

where ρ_{m0} and ρ_{r0} are the densities at present time $t_0 = 0$.

The initial conditions for the differential equations (7.4) are (see (7.10))

$$a(t_0) = h(t_0) = 1, \mathrm{a}'(t_0) = H_0, h'(t_0) = h'_0 \tag{7.12}$$

where H_0 is the Hubble constant and h'_0 is a further constant which does not arise by the use of the theory of general relativity.

Put

$$\varphi_0 = 3H_0\left(1 + \frac{1}{6}\frac{h\prime_0}{H_0}\right) \tag{7.13}$$

Then, it follows from (7.4) and (7.5) by the use of the initial conditions (7.12)

$$\frac{h\prime}{h} = -6\frac{a'}{a} + 2\frac{4\kappa c^4\lambda t + \varphi_0}{2\kappa c^4\lambda t^2 + \varphi_0 t + 1}. \tag{7.14a}$$

Further integration with the initial conditions (7.12) yields

$$a^3\sqrt{h} = 2\kappa c^4\lambda t^2 + \varphi_0 t + 1. \tag{7.14b}$$

Relation (7.8) gives with $t = t_0$

$$\frac{1}{3}(8\kappa c^4\lambda - \varphi_0^2) = 4\left(\frac{8}{3}\pi k\left(\rho_{m0} + \rho_{r0} + \frac{\Lambda c^2}{8\pi k}\right) - H_0^2\right). \tag{7.15}$$

We define as usually the density parameters

$$\Omega_m = \frac{8\pi k\rho_{m0}}{3H_0^2}, \Omega_r = \frac{8\pi k\rho_{r0}}{3H_0^2}, \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{7.16}$$

and put

$$\Omega_m K_0 = \Omega_m + \Omega_r + \Omega_\Lambda - 1. \tag{7.17}$$

It follows from (7.8) by the use of (7.14), (7.17) and the elimination of h and h' in (7.8) the differential equation

$$\left(\frac{a'}{a}\right)^2 = \frac{H_0^2}{(2\kappa c^4\lambda t^2 + \varphi_0 t + 1)^2}\{-\Omega_m K_0 + \Omega_r a^2 + \Omega_m a^3 + \Omega_\Lambda a^6\}. \tag{7.18}$$

Relation (7.15) is rewritten by the use of (7.16) and (7.17):

$$\frac{2\kappa c^4\lambda}{H_0^2} - \left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 = 3\Omega_m K_0. \tag{7.19}$$

It follows from (7.19) that

$$K_0 > 0 \tag{7.20}$$

is equivalent to

$$2\kappa c^4\lambda t^2 + \varphi_0 t + 1 > 0 \tag{7.21}$$

for all $t \in]-\infty, +\infty[$. Hence, we have from (7.18) and (7.14b) that (7.20) is necessary and sufficient for the existence of non-singular cosmological models.

Hence, the sum of the density parameters (7.17) is greater than one. Therefore, the solutions $a(t)$ of (7.18) and $h(t)$ given by (7.14b) describe a homogeneous, isotropic model of the universe.

We get from (7.18) that

$$a(t) \geq a_1 > 0 \tag{7.22a}$$

for all $t \in]-\infty, +\infty[$ where a_1 is defined by

$$\Omega_\Lambda a_1^6 + \Omega_m a_1^3 + \Omega_r a_1^2 - \Omega_m K_0 = 0. \tag{7.22b}$$

We require

$$K_0 \ll 1$$

to get small values for $a(t)$.In the following, let us assume

$$\rho_{r0} = 0. \tag{7.23}$$

Under the condition (7.23) an analytic solution of (7.18) can be given. Were write the equation (7.18) with initial condition (7.10) in the form

$$\frac{a\prime}{a} = \pm \frac{H_0}{2\kappa c^4 \lambda t^2 + \varphi_0 t + 1} (-\Omega_m K_0 + \Omega_m a^3 + \Omega_\Lambda a^6)^{1/2}, a(0) = 1. \quad (7.24)$$

The upper (lower) sign implies an increasing (decreasing) of the function $a(t)$. Standard integration methods and some trigonometric addition theorems give after longer calculations for the upper sign the solution

$$a^3(t) = 2K_0 \Big/ \left\{1 - (1 - 2K_0) cos\left(\sqrt{3}\alpha(t)\right) - 2\left(\frac{K_0}{\Omega_m}\right)^{1/2} sin\left(\sqrt{3}\alpha(t)\right)\right\} \quad (7.25a)$$

where

$$\alpha(t) = arctg\left\{(3\Omega_m K_0)^{1/2} H_0 t \Big/ \left(1 + \frac{1}{2}\varphi_0 t\right)\right\}. \quad (7.25b)$$

The detailed derivation of the result (7.25) is found in the article of [Pet 90].

We will now calculate the time t_1 where $a(t)$ reaches its minimal value a_1.This follows from

$$a'(t_1) = 0,$$

i.e., by the use of relation (7.18)

$$(1 - 2K_0) sin\left(\sqrt{3}\alpha(t_1)\right) - 2\left(\frac{K_0}{\Omega_m}\right)^{1/2} cos\left(\sqrt{3}\alpha(t_1)\right) = 0. \quad (7.26)$$

We get by elementary considerations

$$H_0 t_1 = -1 \Big/ \left\{\frac{1}{2}\frac{\varphi_0}{H_0} - \frac{(3\Omega_m K_0)^{1/2}}{A}\right\} \quad (7.27a)$$

with

$$A = tg\left\{\frac{1}{\sqrt{3}}\left(-\pi + arctg\left[2\frac{(K_0/\Omega_m)^{1/2}}{1 - K_0}\right]\right)\right\} \approx 4.03 + O(\sqrt{K_0}). \quad (7.27b)$$

A small value of the function $a(t)$ in the early universe yields

$$0 < K_0 \ll 1. \quad (7.28)$$

To get a solution $a(t)$ which goes to infinity as $t \to \infty$ the denominator of (7.25a) must go to zero. This implies

$$\frac{1}{2}\frac{\varphi_0}{H_0} \approx \frac{3}{2}\Omega_m / \left(1-\sqrt{\Omega_\Lambda}\right) \tag{7.29}$$

where expressions containing K_0 are omitted by virtue of (7.28).

We will now state a simple asymptotic formula for $t \to \infty$.

Define $\tilde{t}_1$ by

$$H_0\tilde{t}_1 = -\frac{1}{2}\frac{\varphi_0}{H_0} / \left(\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 + 3\Omega_m \mathrm{K}_0\right).$$

It follows from (7.25) by longer elementary calculations the asymptotic representation (see e.g. $|Pet\ 98b|$)

$$a^3(t) \approx \frac{3}{4}\left(\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0t - H_0\tilde{t}_1)^2 / \left\{\sqrt{\Omega_\Lambda}(H_0t - H_0\tilde{t}_1) + \frac{1}{3}\frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}\right\}$$

$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}.$$

Let us assume that t_2 is as large such that the above asymptotic formulae hold and denote by τ_2 the corresponding proper time. Then, we get

$$\tau = \tau_2 + \int_{t_2}^{t} \frac{1}{\sqrt{h(t)}} dt \approx \tau_2 + \frac{1}{3\sqrt{\Omega_\Lambda}}\frac{1}{H_0}\log\left\{\frac{\sqrt{h(t)}}{\sqrt{h(t_2)}}\right\}.$$

This relation implies the asymptotic density of matter

$$\rho_m(\tau) = \frac{\rho_{m0}}{\sqrt{h(t)}} \approx \frac{\rho_{m0}}{\sqrt{h(t_2)}} exp\left\{-3\sqrt{\Omega_\Lambda}H_0(\tau-\tau_2)\right\}.$$

Hence, we get an exponential decay of matter in analogy to the radio-active decay.

The case $\Omega_\Lambda = 0$ must be considered separately by virtue of (7.17) with (7.28) and will be studied in sub-chapter 7.2.

The solution of (7.18) with (7.23) and the initial condition $a(t_1) = a_1$ can also be given. It holds

$$a^3(t) = 2a_1^3\left(1 + \frac{\Omega_\Lambda}{\Omega_m}a_1^3\right) \Big/ \left(1 + \left(1 + 2\frac{\Omega_\Lambda}{\Omega_m}a_1^3\right) cos\left(\sqrt{3}\beta(t)\right)\right) \tag{7.30a}$$

for all $t \in \,]-\infty, +\infty[$. Here, it holds

$$\beta(t) = arctg\left\{\frac{\sqrt{3\Omega_m K_0}H_0(t-t_1)}{\left(1+\frac{1}{2}\varphi_0 t_1+\left[\left(\left(\frac{1\varphi_0}{2H_0}\right)^2+3\Omega_m K_0\right)H_0 t_1+\frac{1\varphi_0}{2H_0}\right]H_0 t\right)}\right\} \tag{7.30b}$$

The relations (7.30) yield as $t \to -\infty$

$$a(-\infty) \approx \left(2/\left(1-cos\left(\sqrt{3}\pi\right)\right)\right)^{1/3} a_1 \approx 1.81 a_1. \tag{7.31a}$$

Hence, the function $a(t)$ starts at $t = -\infty$ from 1.81 a_1, decreases to a_1 and then increases to infinity as t goes to infinity. It follows from (7.14) by the use of (7.31a) as $t \to -\infty$:

$$\sqrt{h} \approx \left(\frac{1}{2}\frac{\varphi_0}{H_0}t+1\right)^2 / a^3(-\infty). \tag{7.31b}$$

The function $h(t)$ starts from infinity at $t = -\infty$, decreases to a positive value and then increases to infinity as t goes to infinity.

Therefore, we have for all $t \in \,]-\infty, +\infty[$

$$a(t) \geq a_1 > 0. \tag{7.32}$$

Hence, we have non-singular cosmological models by virtue of (7.11). In the beginning of the universe there are no matter and no vacuum energy, which is given by the cosmological constant, i.e. all the energy is in form of gravitational energy. In the course of time matter and radiation arises at coasts of gravitational energy. After a certain time the energy of matter and of radiation decreases and again go to zero as in the beginning of the universe. Therefore, in contrast to general relativity there is no singularity, i.e. no big bang.

The second law of thermodynamics is given by

$$dU = -PdV + TdS \tag{7.33}$$

where U, P, V, T and S denote energy, pressure, volume, absolute temperature and entropy.

The conservation of the whole energy (7.8) gives with

$$U = C\lambda c^2 \tag{7.34}$$

with a suitable constant C the relation

$$\frac{d}{dt}U = 0. \tag{7.35}$$

The comparison of (7.35) and (7.33) yields

$$PdV = TdS. \tag{7.36}$$

The right hand side of equation (7.36) is by the third law of thermodynamics non-negative. For an expanding space it must hold $P \geq 0$. Here, P is the pressure of the gravitational field and of the field implied by the cosmological constant. It follows by the use of the asymptotic formulae for a^3 and $\sqrt{h}$ that this condition is at least fulfilled for sufficiently large times. In the beginning of the universe space is contracting by virtue of (7.24). Hence, for $t \to -\infty$ it must hold $P \leq 0$. The pressure of the gravitational field dominates the other ones. We get by the use of (7.31):

$$P \approx \frac{1}{2\kappa c^2}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 H_0^2 > 0$$

which contradicts the condition for $t \to -\infty$. These considerations also hold for the case of $\Lambda = 0$ because the cosmological constant is not important in the beginning of the universe.

The application of equation (7.9) to the third law of thermodynamics requires a non-increasing function $h(t)$ to get entropy production in contradiction to the increasing of this function. Hence, we see that for the case $\Lambda > 0$ the universe is not expanding and not contracting and there is no entropy production.

Cosmological models without singularities by the use of flat space-time theory of gravitation are already studied in the article $[Pet\ 81b]$.

7.2 Homogeneous Isotropic Cosmological Model without Cosmological Constant

In this sub-chapter cosmological models with $\Lambda = 0$ and without loss of generality with put $\Omega_r = 0$. We start with the previous section under the assumption $\Omega_\Lambda = 0$. We get byequation (7.22b)

$$K_0 = a_1^3. \tag{7.37}$$

It follows from (7.25) as $t \to \infty$ by longer elementary calculations and the use of (7.28) and (7.13)

$$\frac{h_0{}'}{H_0} = -\frac{2}{3}K_0 < 0. \tag{7.38}$$

Again the considerations of (7.33) to (7.36) hold implying a non-expanding universe. But we may also start from the relation (7.9) by multiplication with a constant C. It follows

$$dU = -\frac{\rho_{r0}}{\left(ah^{1/8}\right)^4} dV - \frac{1}{2} C(\rho_m + \rho_r) \frac{h\prime}{h} dt. \tag{7.39}$$

By the use of the black body temperature

$$T = T_0/\left(ah^{1/8}\right) \tag{7.40}$$

relation (7.39) has again the form of the second law of thermodynamics (7.33) with

$$\begin{aligned} &U = C(\rho_m + \rho_r),\ V = Ca^3,\ \ P = \rho_{r0}\left(\frac{T}{T_0}\right)^4, \\ &dS = -\frac{1}{2} C(\rho_m + \rho_r) \frac{ah^{1/8}}{T_0} \frac{h\prime}{h} dt. \end{aligned} \tag{7.41}$$

Relation (7.41) implies by the use of the second law of thermodynamics that $h(t)$ must decrease for all $t \in]-\infty, +\infty[$ to give entropy production. This is at present time$t_0$stated by (7.38). Hence, in the case $\Lambda = 0$ the interpretation of a contracting and then expanding universe is possible. The time t_1 is again calculated as before by $a'(t_1) = 0$ of the solution (7.25). It follows by longer calculations and condition (7.28):

$$H_0 t_1 \approx -\frac{2}{3}\left\{1 + \frac{3}{2} \frac{\sqrt{3K_0}}{tg\left(-\pi/\sqrt{3}\right)}\right\}. \tag{7.42a}$$

We get by the use of (7.14b) with (7.19) and (7.37)

$$h(t_1)^{1/2} \approx \frac{4}{3}\left(1 + 1/tg^2\left(-\pi/\sqrt{3}\right)\right) \approx 1.41. \tag{7.42b}$$

Therefore, the creation of matter given by (7.11) from the minimal value of $a(t)$ at t_1 till the present time $t_0 = 0$ is given by a factor of about 1.4. It follows from (7.17)

$$\Omega_m = 1/(1 - K_0) \approx 1 + K_0. \tag{7.43}$$

The density parameter of matter Ω_m is a little bit greater than one.

The epoch before t_1, i.e. $t \in]-\infty, t_1[$ can be received by the results of chapter 7.1.

All these results are contained in the articles of Petry [$Pet\ 90a$] and [$Pet\ 90b$].

We will now give an asymptotic solution as $t \to \infty$. Put

$$H_0\tilde{t}_1 = -1/\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right) \tag{7.44a}$$

and assume that

$$H_0 t \gg H_0\tilde{t}_1. \tag{7.44b}$$

Then, the differential equation (7.18) with $\Omega_\Lambda = 0$ has the form

$$\frac{a\prime}{a} \approx H_0\sqrt{\Omega_m}a(t)^{3/2}\Big/\left(\frac{1}{2}\frac{\varphi_0}{H_0}\left(H_0 t - H_0\tilde{t}_1\right)\right)^2. \tag{7.45a}$$

The solution is given by

$$a^3(t) \approx \frac{4}{9}\frac{1}{\Omega_m}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^4\left(H_0 t - H_0\tilde{t}_1\right)^2. \tag{7.45b}$$

The study of $a(t)$ for $t \to \infty$ given by (7.25) and the use of (7.13) and (7.28) imply

$$h(t)^{1/2} \approx \frac{9}{4}\Omega_m/\left(\frac{2\kappa c^4\lambda}{H_0^2}\right) \approx 1 - \frac{1}{9}K_0. \tag{7.46}$$

Hence, the function $h(t)$ is in the above stated region nearly constant and converges to a positive value which is a little bit smaller than one. Therefore, the function $a(t)$ starts from a finite positive value $a(-\infty)$ and $h(t)$ from infinity. In the course of time $a(t)$ decreases till to the time t_1 with a value $a(t_1) = a_1 > 0$. After that time $a(t)$ always increases to infinity. The function $h(t)$ decreases for all times to a positive value which is a little bit smaller than the present value in agreement with the third law of thermodynamics stated by relation (7.9). Hence, in the case that the cosmological constant is zero the interpretation of an expanding space is permitted but also the interpretation of a non-expanding space is possible by the considerations of sub-section 7.1.

Furthermore, it appears that at present time the universe is nearly stationary. In the final state only matter and gravitational energy exist.

It is worth mentioning that in addition to matter, radiation and cosmological constant a further kind of energy may be introduced in the study of cosmological models. Such considerations can be found in the articles [$Pet\ 94c$] and [$Pet\ 98a$].

A cosmological model with a scaling dependent cosmological constant is studied in the article [Pet 08].

Summarizing, the cosmological models of flat space-time theory of gravitation are in the beginning of the universe quite different from those of general relativity. In the beginning strong gravitational fields exist. The received models are non-singular, i.e. a big bang does not exist. Formula (7.45b) is identical with the result of general relativity, i.e. for sufficiently large times after the minimum of the universe the two theories give the same result.

Chapter 8

Expanding or Non-Expanding Universe

8.1 Non-Expanding Universe

In this sub-chapter we will study non-expanding universes. It is well-known that the observed redshift of distant objects (galaxies, quasars) are interpreted as Doppler-effect, i.e. the observed universe is expanding. Furthermore, astrophysical observations indicate an accelerated expansion in the recent epoch.

In the previous chapter we have shown that an expanding universe can be received if the cosmological constant $\Lambda = 0$. This result can be used to explain the observed redshift of distant objects. If the cosmological constant $\Lambda \neq 0$ the third law of thermodynamics is violated. Hence, another form of the second law of Thermodynamics containing the whole energy of the universe is considered. This law implies that there are no expansion and no contraction of the universe and entropy is not produced in the course of time.

It is worth to mention that this law is also applicable in the special case $\Lambda = 0$.

8.2 Proper Time and Absolute Time

In addition to the system time t and the proper time τ in the previous chapters we define the absolute time t'.

The proper time $\tilde{\tau}$ for an object at rest is defined by

$$d\tilde{\tau} = \frac{1}{\sqrt{h(t)}} dt. \tag{8.1}$$

This gives for the whole proper time since the beginning of the universe

$$\tilde{\tau}(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{h(t)}} dt. \tag{8.2}$$

The proper time is used in the study of general relativity.

In the beginning of the universe we have

$$a(t) \approx a(-\infty) > 0.$$

This implies by the use of (7.14b) that

$$h^{1/2}(t) \approx \frac{1}{\left(a(-\infty)\right)^3} \frac{2\kappa c^4 \lambda}{H_0^2} (H_0 t)^2 \tag{8.3}$$

for $t \to -\infty$. Hence, relation (8.2) implies the proper time $\tilde{\tau}(t)$ at any time t. The equation (7.18) can be written by the use of (7.14b)

$$\left(\frac{1}{a}\frac{da}{d\tilde{\tau}}\right)^2 = H_0^2\left(-\frac{\Omega_m K_0}{a^6} + \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda\right). \tag{8.4}$$

The differential equation (8.4) is for $K_0 = 0$ identical with the equation given by general relativity. The case $K_0 = 0$ implies the singularity, i.e., the big bang by Einstein's theory. But K_0 must be greater than zero and must fulfil the condition (7.28) which avoids the singularity. Hence, relation (8.4) implies that for $a(t)$not too small that the result of flat space-time theory of gravitation agrees with that of general relativity, i.e. shortly after the big bang of Einstein's theory.

We will now introduce the absolute time t' by

$$dt' = \frac{1}{a(t)\sqrt{h(t)}}dt = \frac{1}{a(t)}d\tilde{\tau}. \tag{8.5}$$

This gives for the proper τ in the universe

$$(cd\tau)^2 = -a^2(t)\{|dx|^2 - (dct')^2\}. \tag{8.6}$$

Relation (8.6) implies for the absolute value of the light-velocity v_L:

$$|v_L| = \left|\frac{dx}{dt'}\right| = c. \tag{8.7}$$

Therefore, the absolute value of the light velocity in the universe is always the vacuum light velocity c. This is the reason that t' is denoted as absolute time. In the further study we will remark that the time t' has advantages relative to the use of the proper time $\tilde{\tau}$ although the proper time is measured by atomic clocks.

The equation (8.4) can be written by the use of (8.5) in the form

$$\left(\frac{da}{dt'}\right)^2 = \frac{H_0^2}{a^2}(-\Omega_m K_0 + \Omega_r a^2 + \Omega_m a^3 + \Omega_\Lambda a^6). \tag{8.8}$$

Furthermore, assume that a light ray is emitted at distance r at time t'_e resp. at time $t'_e + dt'_e$ and it is received by the observer at time $t' = 0$ resp. at time $0 + dt'$. Then, it holds by the use of (7.8)

$$r = c\int_{t'_e}^{0} dt' = -c\,t'_e,\ r = c\int_{t'_e+dt'_e}^{dt'} dt' = c(dt' - t'_e - dt'_e).$$

These two relations give

$$dt' = dt'_e.$$

This is a further reason that t' is the absolute time.

8.3 Redshift

We will now calculate the frequency emitted from a distant object at rest and received by the observer at rest at present time. The use of the absolute time t' simplifies the calculation although the use of the system time t and of the proper time $\tilde{\tau}$ would give the same result.

Let us assume that an atom at rest in a distant object emits a photon at time t'_e. The proper time is by virtue of (8.6):

$$d\tau = a(t'_e)dt'. \tag{8.9}$$

The energy of the emitted photon is

$$E \sim -g_{44}(t'_e)\frac{dt\prime}{d\tau} \sim a(t'_e)E_0. \tag{8.10}$$

The photon moves to the observer and it arrives at time $t' = t'_0$. Let (p_1, p_2, p_3, p_4) be the four-momentum of the photon in the universe with

$$p_4 = -E(t')/c.$$

Then, it follows from equation (1.30) with $i = 4$ by the use of (8.6) and (8.7)

$$\frac{d}{dt\prime}\left(g_{44}\frac{dt\prime}{d\tau}\right) = a(t')\frac{da}{dt\prime}(c^2 - |v_L|^2) = 0, \tag{8.11}$$

i.e., the energy of the emitted photon is constant during its motion. It is worth to mention that the conservation of the energy of the photon during its motion to the observer only holds by the use of the absolute time t'. Hence, we have by the law

$$E = h\nu$$

where here h denotes the Planck constant that the arriving photon has the frequency

$$\nu = a(t'_e)\nu_0. \tag{8.12}$$

Here, ν_0 is the frequency emitted by the same atom at rest and at present time. This gives the reshift

$$z = \frac{\nu_0}{\nu} - 1 = \frac{1}{a(t'_e)} - 1. \tag{8.13}$$

This redshift formula is also received by the use of the proper time $\tilde{\tau}$ and the system time t. This results can be found in the article of Petry [Pet 08].

We will now give the distance-redshift relation. Equation (8.6) implies for light emitted at distance r at time t'_e and received at $r = 0$ at time t'_0 by the use of (8.7)

$$r = c \int_{t'_e}^{t'_0} dt' = c(t'_0 - t'_e). \tag{8.14}$$

Equation (8.8) yields by differentiation

$$2a \frac{da}{dt'} \left(-\frac{1}{a^2} \left(\frac{da}{dt'} \right)^2 + \frac{1}{a} \frac{d^2 a}{dt'^2} \right) = H_0^2 \left(4 \frac{\Omega_m K_0}{a^5} - 2 \frac{\Omega_r}{a^3} - \frac{\Omega_m}{a^2} + 2\Omega_\Lambda \right) \frac{da}{dt'}$$

This relation gives at present time t'_0:

$$\frac{d^2 a(t'_0)}{dt'^2_0} = H_0^2 \left(1 + 2\Omega_m K_0 - \Omega_r - \frac{1}{2}\Omega_m + \Omega_\Lambda \right). \tag{8.15}$$

The redshift (8.13) is approximated by Taylor expansion and the use of (8.14)

$$z \approx H_0 \frac{r}{c} + \left(1 - \frac{1}{2} \frac{1}{H_0^2} \frac{d^2 a(t'_0)}{dt'^2_0} \right) \left(H_0 \frac{r}{c} \right)^2.$$

Hence, we get by (8.15) and (7.17) and neglecting small expressions the redshift:

$$z \approx H_0 \frac{r}{c} + \frac{3}{4} \Omega_m \left(H_0 \frac{r}{c} \right)^2. \tag{8.16}$$

We easily get the redshift formula to higher order by the use of Taylor expansion to higher order. By virtue of the use of the absolute time t' only differentiation to higher order of (8.8) are needed by virtue of (8.14).

For an expanding universe the redshift follows by the transformation

$$X^i = a(\tilde{\tau}) x^i \quad \text{(i=1,2,3)} \tag{8.17a}$$

with the velocity

$$\frac{d}{d\tilde{\tau}} X^i = \frac{da(\tilde{\tau})}{d\tilde{\tau}} x^i \quad \text{(i=1,2,3).} \tag{8.17b}$$

The proper time τ of the universe with the coordinates (X^1, X^2, X^3) and the proper time $\tilde{\tau}$ is given by:

$$
\begin{aligned}
(cd\tau)^2 = &-\sum_{k=1}^{3}(dX^k)^2 + \frac{2}{c}\sum_{k=1}^{3}\frac{1}{a}\frac{da}{d\tilde{\tau}}X^k dX^k d(c\tilde{\tau}) \\
&+(dc\tilde{\tau})^2\left(1-\left(\frac{1}{c}\frac{1}{a}\frac{da}{d\tilde{\tau}}|X|\right)^2\right).
\end{aligned}
\tag{8.18}
$$

Relation (8.18) implies that locally, i.e. $X^i = 0$ (i=1,2,3) the velocity of light is equal to the vacuumlight velocity. This result is connected with the ideas of Einstein that locally the pseudo-Euclidean geometry holds. We get by the substitution of (8.17) into relation (8.18)

$$d\tau = d\tilde{\tau},$$

i.e. any observer in the expanding universe is given by (8.18) and has the proper-time $\tilde{\tau}$. The theory of gravitation in flat space-time doesn't use (8.18) and it is therefore not further studied.

We will mention that a universe which at first contracts and then expands has no singularity, i.e. there is no big bang. Instead of the big bang we have a universe with a bounce.

In flat space-time theory of gravitation the redshift may be explained without expansion of space by the conservation of the whole energy (7.8) of the universe. It follows from (7.7) with (7.11) that the different kinds of matter, of radiation and of gravitation are transformed into one another in the course of time by the time-dependence of a and h. This is in analogy to the result that the gravitational field influences the redshift(see (2.69)).

There sults of this chapter can be found in several articles of Petry [*Pet* 97*b*, *98a, 98b, 02, 08, 11a, 13b*.

8.4 Age of the Universe

We will now calculate the age of the universe measured with absolute timet'. It follows by the use of (8.8) for the age after the minimum of the function$a(t')$ till the present time:

$$\begin{aligned} T_{t'}(t'_0) &= \int_{t'_1}^{t'_0} dt' = \int_{a_1}^{1} 1/\left(\frac{da}{dt'}\right) da \\ &= \frac{1}{H_0}\int_{a_1}^{1} \frac{a da}{(-\Omega_m K_0+\Omega_r a^2+\Omega_m a^3+\Omega_\Lambda a^6)^{1/2}} \\ &\geq \frac{1}{H_0}\int_{a_1}^{1} \frac{a da}{(-\Omega_m K_0+(\Omega_r+\Omega_m+\Omega_\Lambda)a^2)^{1/2}} \\ &= \frac{1}{H_0}\left(1-(-\Omega_m K_0+(1+\Omega_m K_0)a_1^2)^{1/2}\right) \approx \frac{1}{H_0}. \end{aligned} \tag{8.19}$$

Therefore, the age of the universe measured with absolute time is greater than $\frac{1}{H_0}$ independent of the density parameters, i.e. there is no age-problem. It seems that the use of the absolute time instead of the proper time is more natural. This is implied by the fact that the time difference at a distant object stated by two different events is measured by the observer at present time with the same value of time difference and the velocity of light is everywhere and at any time equal to the vacuum light velocity.

Summarizing: Flat space-time theory of gravitation gives cosmological models with bounce and without big bang. Furthermore, the models can be interpreted as non-expanding universe. The redshift is explained by the transformation of the different kindsof energy into one another in the course of time whereas the whole energy of the universe is conserved. This interpretation can also be found in the article of Petry [Pet 07]. It follows that the introduction of the absolute time t' simplifies the computations. The expansion of space was at earlier times the only interpretation of the redshift. In the meantime there are many authors who negate the expansion and assume that the redshift is intrinsic.

Chapter 9

Perturbations in the Universe

In this chapter the theory of linear perturbations in the universe are studied.

9.1 Differential Equations of Linear Perturbation in the Universe

A covariant, linear, cosmological perturbation theory is given. The metric is the pseudo-Euclidean geometry. The energy-momentum tensor is stated and the basic equations for the propagation of the perturbations are presented. The perturbed equations for a homogeneous isotropic universe are stated. All the results of this chapter can be found in [$Pet\ 95b$].

We use the pseudo-Euclidean geometry (1.5), the theory of gravitation in flat space-time (1.23), the equations of motion (1.29) and the conservation of the whole energy-momentum (1.25). The matter tensor is given by (1.28).

The gravitational field satisfies

$$g^{ij} \to g^{ij} + \Delta g^{ij} \tag{9.1a}$$

with the condition

$$|\Delta g^{ij}| \ll |g^{ij}|. \tag{9.1b}$$

It follows by linear perturbation

$$g_{ij} \to g_{ij} + \Delta g_{ij} \tag{9.2a}$$

with the result

$$\Delta g_{ij} = -g_{ik} g_{jl} \Delta g^{kl}. \tag{9.2b}$$

In addition, we put

$$\rho \to \rho + \Delta\rho, p \to p + \Delta p, u^i \to u^i + \Delta u^i. \tag{9.3}$$

The arising equations of perturbations are applied to cosmological models with

$$\begin{aligned} g^{ij} &= 1/a^2(t) \quad (i = j = 1,2,3) \\ &= -h(t) \quad (i = j = 4) \\ &= 0 \quad (i \neq j) \end{aligned}$$

as considered in chapter VII.

Let $(\Delta v^1, \Delta v^2, \Delta v^3)$ be the perturbed velocity. Put the perturbed potentials

$$f = a^2 \Delta g^{ii}, d = -\frac{1}{h}\Delta g^{44}, b_i = a^2 \Delta g^{i4} \quad \text{(i=1,2,3)}. \tag{9.4}$$

Then, the cosmological model implies after longer calculations the differential equations for the perturbed field

(1) $$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial f}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial f}{\partial t}\right) = -\frac{2}{c}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{a'}{a}b_k\right)$$
$$-\frac{3}{c^2}a^3\sqrt{h}\frac{a'}{a}\left(\frac{\partial f}{\partial t} + \frac{\partial d}{\partial t}\right) - 3\kappa c^2(\rho - p)(f + d) + 2\kappa c^2(\Delta\rho - \Delta p)$$

(2) $$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial b_i}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial b_i}{\partial t}\right) = -\frac{1}{c^2}a^3\sqrt{h}\frac{\partial b_i}{\partial t}\left(2\frac{a'}{a} + \frac{h'}{h}\right)$$
$$-\frac{1}{c^2}a^3\sqrt{h}b_i\left(-3\left(\frac{a'}{a}\right)^2 + 3\frac{a'}{a}\frac{h'}{h} + \frac{1}{4}\left(\frac{h'}{h}\right)^2\right)$$
$$+\frac{3}{2}\frac{1}{c}a^3\sqrt{h}\left(-\frac{a'}{a} + \frac{1}{2}\frac{h'}{h}\right)\frac{\partial f}{\partial x^i} - \frac{1}{2}\frac{1}{c}a^3\sqrt{h}\left(3\frac{a'}{a} + \frac{1}{2}\frac{h'}{h}\right)\frac{\partial d}{\partial x^i} +$$
$$4\kappa c^2(\rho + p)a^2 h\frac{\Delta v^i}{c} \quad \text{(i=1,2,3)}$$

(3) $$\sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{\partial d}{\partial x^k}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial d}{\partial t}\right)$$
$$= -\frac{1}{c}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\frac{a}{\sqrt{h}}\frac{h'}{h}b_k\right)$$
$$+\frac{3}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a} + \frac{h'}{h}\right)\frac{\partial f}{\partial t} + \frac{1}{c^2}a^3\sqrt{h}\left(-3\frac{a'}{a} + \frac{h'}{h}\right)\frac{\partial d}{\partial t}$$
$$+3\kappa c^2(\rho + 3p)(f + d) - 2\kappa c^2(\Delta\rho + 3\Delta p). \tag{9.5}$$

The perturbed equations of motion are:

(1) $$\frac{\partial \Delta p}{\partial x^i} + \frac{1}{c}\frac{\partial}{\partial t}\left\{(\rho + p)\left(b_i + a^2 h\frac{\Delta v^i}{c}\right)\right\} = -\frac{3}{2}p\frac{\partial f}{\partial x^i} + \frac{1}{2}p\frac{\partial d}{\partial x^i} \quad \text{(i=1,2,3)}$$

(2) $$-(\rho + p)\sum_{k=1}^{3}\frac{\partial \Delta v^k}{\partial x^k} - \frac{\partial \Delta p}{\partial t} = -\frac{3}{2}p\frac{\partial f}{\partial t} + \frac{1}{2}\rho\frac{\partial d}{\partial t} + 3\frac{a'}{a}\Delta p + \frac{1}{2}\frac{h'}{h}\Delta\rho. \tag{9.6}$$

Furthermore, we have an equation of state for the perturbed pressure. i.e.,

$$\Delta p = p(\Delta\rho). \tag{9.7}$$

The relations (9.5), (9.6) and (9.7) are ten equations for the ten unknown functions f, b_i (i=1,2,3), $d, \Delta v^i$ (i=1,2,3), Δp and $\Delta\rho$. These equations describe small perturbations in a homogeneous, isotropic cosmological model.

In the following let us assume that the equation of state has the form

$$\Delta p = v_s^2 \Delta\rho \tag{9.8}$$

with constant velocity sound $0 \leq v_s \leq 1$. It follows as consequence of the perturbed field equation (9.5) and the perturbed equations of motion (9.6) a conservation law of the perturbed energy-momentum tensor (see $[Pet\ 95b]$).

9.2 Spherically Symmetric Perturbations

We will now study spherically symmetric solutions of the perturbed equations (9.5), (9.6) and (9.8). The study of these results are contained in the following sub-chapters and are found in the articles $[Pet\ 96a]$ and $[Pet\ 96b]$.

Let r denote the Euclidean distance from the centre of the spherical symmetry and let k be the wave number. We make the ansatz

$$\begin{gathered} f(r,t) \to \tilde{f}(k,t)\frac{sin(kr)}{r},\ d(r,t) \to \tilde{d}(k,t)\frac{sin(kr)}{r}, \\ b_i(r,t) \to \frac{c}{H_0}\tilde{b}(k,t)\frac{\partial}{\partial x^i}\left(\frac{sin(kr)}{r}\right), \\ \Delta\rho(r,t) \to \tilde{\rho}(k,r)\frac{sin(kr)}{r}, \Delta v^i(r,t) \to \frac{c}{H_0}\tilde{v}(k,t)\frac{\partial}{\partial x^i}\left(\frac{sin(kr)}{r}\right). \end{gathered} \tag{9.9}$$

We get by substituting the relations (9.9) into the equations (9.5) and (9.6)

(1) $$-k^2\frac{a}{\sqrt{h}}\tilde{f} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{f}}{\partial t}\right) = -\frac{2}{c}k^2\frac{a}{\sqrt{h}}\frac{a'}{a}\frac{c}{H_0}\tilde{b} - \frac{3}{c^2}a^3\sqrt{h}\frac{a'}{a}\left(\frac{\partial f}{\partial t} + \frac{\partial\tilde{d}}{\partial t}\right)$$
$$-3\kappa c^2(\rho - p)\left(\tilde{f} + \tilde{d}\right) + 2\kappa c^2(1 - v_s^2)\tilde{\rho},$$

(2) $$-k^2\frac{a}{\sqrt{h}}\tilde{b} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{b}}{\partial t}\right) = -\frac{1}{c^2}a^3\sqrt{h}\frac{\partial\tilde{b}}{\partial t}\left(2\frac{a'}{a} + \frac{h'}{h}\right)$$
$$-\frac{1}{c^2}a^3\sqrt{h}\tilde{b}\left(-3\left(\frac{a'}{a}\right)^2 + 3\frac{a'}{a}\frac{h'}{h} + \frac{1}{4}\left(\frac{h'}{h}\right)^2\right)$$

$$+\frac{3}{2}\frac{H_0}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{f}-\frac{1}{2}\frac{H_0}{c^2}a^3\sqrt{h}\left(3\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{d}$$
$$+4\kappa c^2(\rho+p)a^2\sqrt{h}\frac{\tilde{v}}{c},$$

(3)
$$-k^2\frac{a}{\sqrt{h}}\tilde{d}-\frac{1}{c^2}\frac{\partial}{\partial t}\left(a^3\sqrt{h}\frac{\partial\tilde{d}}{\partial t}\right)=\frac{1}{c}k^2\frac{a}{\sqrt{h}}\frac{h'}{h}\frac{c}{H_0}\tilde{b}$$
$$+\frac{3}{c^2}a^3\sqrt{h}\left(-\frac{a'}{a}+\frac{h'}{h}\right)\frac{\partial\tilde{f}}{\partial t}+\frac{1}{c^2}a^3\sqrt{h}\left(-3\frac{a'}{a}+\frac{h'}{h}\right)\frac{\partial\tilde{d}}{\partial t} \tag{9.10}$$
$$+3\kappa c^2(\rho+3p)\big(\tilde{f}+\tilde{d}\big)-2\kappa c^2(1+3v_s^2)\tilde{\rho}.$$

(4)
$$v_s^2\tilde{\rho}+\frac{1}{c}\frac{\partial}{\partial t}\left\{(\rho+p)\frac{c}{H_0}\left(\tilde{b}+a^2h\frac{\tilde{v}}{c}\right)\right\}=-\frac{3}{2}p\tilde{f}+\frac{1}{2}\rho\tilde{d},$$

(5)
$$(\rho+p)k^2\frac{c^2}{H_0}\frac{\tilde{v}}{c}-\frac{\partial\tilde{\rho}}{\partial t}=-\frac{3}{2}p\frac{\partial\tilde{f}}{\partial t}+\frac{1}{2}\rho\frac{\partial\tilde{d}}{\partial t}+\left(3v_s^2\frac{a'}{a}+\frac{1}{2}\frac{h'}{h}\right)\tilde{\rho}. \tag{9.11}$$

The three perturbed field equations (9.10) and the two perturbed equations of motion (9.11) are five linear homogeneous differential equations for the five unknown functions $\tilde{f},\tilde{b},\tilde{d},\tilde{\rho},\tilde{v}$ depending on t and on a parameter k. Knowing a solution of (9.10) and (9.11) for k on a fixed interval I we can get a more general solution by virtue of the linearity of the equations. Let $B(k)$ be a function of k on the interval I and let r_0 be a fixed distance from the centre then we get the more general solutions

$$\begin{aligned}
f(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0^2\tilde{f}(k,t)dk,\\
b_i(r,t)&=\int B(k)\frac{\partial}{\partial x^i}\left(\frac{\sin(kr)}{r}\right)r_0^2\frac{c}{H_0}\tilde{b}(k,t)dk,\\
d(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0^2\tilde{d}(k,t)dk,\\
\Delta\rho(r,t)&=\int B(k)\frac{\sin(kr)}{r}r_0\tilde{\rho}(k,t)dk,\\
\Delta v^i(r,t)&=\int B(k)\frac{\partial}{\partial x^i}\left(\frac{\sin(kr)}{r}\right)r_0\frac{c}{H_0}\tilde{v}(k,t)dk.
\end{aligned} \tag{9.12}$$

Here, the integration is taken over the interval I.

In the following we will only consider cosmological models with $\Lambda=0$, i.e.

$$\Omega_m\approx 1 \tag{9.13}$$

and the case

$$v_s^2 = 0. \tag{9.14}$$

We put

$$\varepsilon = \frac{\rho_{r0}}{\rho_{m0}}. \tag{9.15}$$

9.3 Beginning of the Universe

The beginning of the universe in flat space-time theory of gravitation is non-singular. All the energy is in form of gravitational energy and radiation and dust arise out of gravitational energy whereas the whole energy is conserved. Put for $t \to -\infty$

$$\xi = +\frac{3}{2}H_0 t - 1. \tag{9.16}$$

Then, we have by (7.31) and (7.14b)

$$\begin{aligned} &a(-\infty) \approx 1.81 a_1, \sqrt{h(t)} \approx \xi^2 / a^3(-\infty), \\ &\rho(t) \approx \rho_{r0} a^2(-\infty)/\xi^2, \ p(t) \approx \frac{1}{3}\rho_{r0} a^2(-\infty)\Big/\xi^2. \end{aligned} \tag{9.17}$$

In the beginning of the universe the density of matter is negligible and only the density of radiation and its pressure dominate.

We make the ansatz

$$\begin{aligned} &\tilde{f} = \sum_{k=0}^{\infty} \frac{f_{2k}}{\xi^{2k+l}}, \tilde{d} = \sum_{k=0}^{\infty} \frac{d_{2k}}{\xi^{2k+l}}, \tilde{\rho}/\rho_{m0} = \sum_{k=0}^{\infty} \frac{\rho_{2k}}{\xi^{2k+l+2}}, \\ &\tilde{b} \ = \sum_{k=0}^{\infty} \frac{b_{2k}}{\xi^{2k+l-1}}, \ \frac{\tilde{v}}{c} = a^4(-\infty) \sum_{k=0}^{\infty} \frac{v_{2k}}{\xi^{2k+l+3}}. \end{aligned} \tag{9.18}$$

We get by the substitution of the relations (9.16), (9.17) and (9.18) into the equations (9.10) and (9.11) and by the use of (9.13), (9.14) and (9.15) five homogeneous linear equations to determine l such that not all of the five coefficients $f_0, b_0, d_0, \rho_0, v_0$ vanish. There exist four non-negative values of l:

$$l = 0, 1, 2, 5. \tag{9.19}$$

In the following only the case $l = 0$ is studied implying two arbitrary constants f_0 and b_2. We get

$$b_0 = 0, d_0 = 3 f_0, v_0 = \frac{1}{2} f_0,$$
$$\rho_0 = 3 f_0 a^2(-\infty)\left(\varepsilon - \frac{3}{10}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right). \tag{9.20a}$$

Furthermore, it follows

$$f_2 = f_0 a^2(-\infty)\left(\frac{2}{3}\varepsilon + \frac{1}{10}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right),$$
$$d_2 = f_0 a^2(-\infty)\left(-\frac{2}{3}\varepsilon + \frac{3}{10}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right),$$
$$v_2 = -b_2 + \frac{1}{12} f_0 a^2(-\infty)\left(-\frac{4}{3}\varepsilon + \frac{1}{5}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right), \tag{9.20b}$$
$$\rho_2 = \frac{2}{3}\varepsilon f_0 a^4(-\infty)\left(\varepsilon + \frac{3}{5}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right).$$

All the other coefficients can be recursively calculated. Hence, we get for $l = 0$ the solution (9.18) depending on two arbitrary parameters. Then, the relations (9.12) give the perturbed solutions in the beginning of the universe. Let us now discuss the received perturbed solution. Assuming $B(k) \geq 0$ on the interval I we have to put $f_0 > 0$ to get collapsing spherically symmetric perturbations in the neighbourhood of the centre as t increases from $-\infty$. This result follows by the use of (9.20a), (9.18) and (9.12). Furthermore, we get that the density of the spherically symmetric perturbation is positive if the wave numbers fulfil the condition

$$0 \leq k \leq \left(\frac{10}{3}\varepsilon\right)^{1/2} \frac{3H_0}{2c} \frac{1}{a(-\infty)}. \tag{9.21}$$

Hence, we have to the lowest order of the density fluctuations as $t \to -\infty$

$$\Delta\rho(r,t)/\rho_m(t) \approx \int B(k) \frac{\sin(kr)}{r} r_0^2 3 \frac{f_0}{a(-\infty)} \left(\varepsilon - \frac{3}{10}\left(\frac{2ck}{3H_0} a(-\infty) \right)^2 \right) dk. \tag{9.22}$$

Therefore, for the case $l = 0$ small spherically symmetric non-homogeneities in the uniform distribution of matter can exist in the beginning of the universe.

The cases $l = 1, 2, 5$ give only one-parametric solutions with

$$\Delta\rho(r,t)/\rho_m(t) = O\left(\frac{1}{\xi^2}\right)$$

for $t \to -\infty$.

Therefore, small non-homogeneities can arise in the homogeneous distribution of matter in the beginning. By virtue of the small horizons there are many unconnected regions in the universe. The non-homogeneities are unconnected and arise independently from one another. Therefore, they are uniformly distributed in space in the beginning of the universe. This may explain the presently observed homogeneity of matter on large scales in the universe. The horizons increase in the course of time and larger regions of the universe become connected. The non-homogeneities are then connected and influence one another by gravitation.

9.4 Matter Dominated Universe

In this sub-chapter the universe is considered where matter dominates radiation. Put

$$\xi = \left(\frac{3}{2} H_0 (t - t_1)\right)^{1/3} \tag{9.23}$$

with

$$H_0 t_1 \approx -\frac{2}{3}. \tag{9.24}$$

During the studied time epoch it holds by (7.46) and (7.14b)

$$h(t) \approx 1, \;\; a(t) \approx \xi^2, \;\; \rho(t) \approx \rho_{m0}, \;\;\; p \approx 0. \tag{9.25}$$

We make the ansatz

$$\tilde{f} = \xi^l \sum_{k=0}^{\infty} f_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \;\; \tilde{d} = \xi^l \sum_{k=0}^{\infty} d_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k},$$

$$\tilde{b} = \xi^{l+3} \sum_{k=0}^{\infty} b_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \;\; \frac{\tilde{\rho}}{\rho_{m0}} = \xi^l \int_{k=0}^{\infty} \rho_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}, \tag{9.26}$$

$$\frac{\tilde{v}}{c} = \xi^{l-1} \sum_{k=0}^{\infty} v_{2k} \left(\frac{ck}{H_0}\xi\right)^{2k}.$$

It follows in analogy to the previous sub-chapter

$$l = 0, 9 \tag{9.27}$$

which imply non-vanishing solutions. Furthermore, a pair of complex numbers is received to get non-vanishing solutions. The case

$$l = 9 \tag{9.28}$$

is further studied. It follows with an arbitrary parameter ρ_0:

$$f_0 = -\frac{13}{3}\rho_0, \ d_0 = -2\rho_0, \ b_0 = -\frac{17}{60}\rho_0, \ v_0 = \frac{7}{60}\rho_0. \tag{9.29a}$$

The coefficients of higher order can again be recursively calculated. It can be proved that the series (9.26) converge absolutely and uniformly. Hence, the sums and the integrals of (9.12) can be exchanged. Put $I = [0, k_0]$ with k_0 sufficiently small, i.e. large scale non-homogeneities we have to the lowest order

$$\Delta\rho(r,t)/\rho_m(t) \approx \rho_0 r_0^2 a(t)^{9/2} \int B(k) \frac{sin(kr)}{r} dk. \tag{9.29b}$$

This solution is non-singular for $r = 0$ whereas in [Isr 94] spherically symmetric perturbations are considered by the use of general relativity yielding a singularity at $r = 0$. Hence, the density contrast in the matter dominated universe increases faster than by the use of general relativity (see e.g. [Bar 80], [Isr 94]). In these articles it is proved that the density contrast increases at most linearly with the function $a(t)$.

Let t_d be the time of the decoupling of matter and radiation. Then, relation (9.29b) yields

$$\Delta\rho(r,t)/\rho_m(t) \approx \Delta\rho(r,t_d)/\rho_m(t_d) \left(\frac{a(t)}{a(t_d)}\right)^{9/2}. \tag{9.30}$$

It holds for adiabatic perturbations

$$|\Delta\rho(r,t_d)/\rho_m(t_d)| \approx 3|\Delta T/T|_d. \tag{9.31}$$

Here, ΔT denotes the temperature anisotropy of CMBR. The decoupling occurs at a redshift z_d (see e.g.,[Kol 90])

$$1/a(t_d) = 1 + z_d \approx 1100. \tag{9.32}$$

The analysis of COBE-data show that the CMBR has an anisotropy of

$$|\Delta T/T|_d \approx 10^{-5} \tag{9.33}$$

on large scales [Smo 92]. Hence, relation (9.30) gives by the use of (9.31), (9.32) and (9.33)

$$|\Delta\rho(r,t)/\rho_m(t)| \approx 3 \cdot 10^{-5}\big(1100 \cdot a(t)\big)^{9/2}. \qquad (9.34)$$

The time $\bar{t}$ where the density contrast is given by

$$|\Delta\rho(r,\bar{t})/\rho_m(\bar{t})| \approx 1$$

implies a redshift $\bar{z}$ with

$$1 + \bar{z} = 1/a(\bar{t}) \approx 1100 \cdot \left(\frac{3}{10^5}\right)^{2/9} \approx 108. \qquad (9.35)$$

Summarizing, large scale structures can arise in the matter dominated universe in accordance with the observed CMBR anisotropy. It is worth to mention that for a density contrast greater than one non-linear perturbations must be considered.

All these results with detailed calculations are given in the articles of [Pet 95a] and [Pet 96a, 96b] where also further remarks can be found.

Spherically symmetric perturbations in a universe which contains an additional field as source are studied in the article [Pet 97a].

In the paper [Gro 97] higher order approximations of density perturbations are given as well in the beginning as in the matter dominated universe. The results are based on numerical computations. Numerical computations of spherically symmetric density perturbations in a universe with an additional field are stated in the paper [Sch 97].

For the study of the early universe and structure formation by the use of Einstein's theory, e.g., the books of [Kol 90] [Pee 80] and [Pad 93] shall be considered.

It should also be remarked that the theory of Einstein implies a too small density contrast which yields difficulties to explain the large scale structures in the universe as galaxies, etc.

Chapter 10

Post-Newtonian Approximation in the Universe

In chapter V the post-Newtonian approximation neglecting the universe, i.e. in empty space is given. In this chapter post-Newtonian approximation in the universe is studied. Here, we follow along the lines of article [Pet 00].

10.1 Post-Newtonian Approximation

The metric is the pseudo-Euclidean geometry given by equation (1.1) with (1.5). The potentials are:

$$\begin{aligned} g_{ij} &= a^2\left(1+\frac{2}{c^2}U\right)\delta_{ij},\ (i;j=1,2,3) \\ &= -\frac{4}{c^2}\frac{a}{\sqrt{h}}V_i,\ (i=1,2,3;\ j=4) \\ &= -\frac{4}{c^2}\frac{a}{\sqrt{h}}V_j,\ (i=4;\ j=1,2,3) \\ &= -\frac{1}{h}\left(1-\frac{2}{c^2}U+\frac{1}{c^4}S\right).(i=j=4) \end{aligned} \tag{10.1}$$

The matter tensor is given by (5.1a) without the factor $\left(\frac{-G}{-\eta}\right)^{1/2}$ in contrast to the considerations of chapter V. In the following, we follow along the lines of chapter V. Let us assume condition (5.5) and (5.6b). Again we get (5.6d). Relation (1.12) with (1.13) implies by the use of (10.1)

$$\frac{dt}{d\tau} = 1+\frac{1}{c^2}U+\frac{1}{2c^2}a^2h|v|^2. \tag{10.2}$$

The matter tensor has to post-Newtonian accuracy the form

$$\begin{aligned} T(M)^i_j &= a^2\rho h v^i v^j + pc^2\delta_{ij},\ (i;j=1,2,3) \\ &= a^2\rho h c v^i\left(1+\frac{\Pi}{c^2}+\frac{p}{\rho}+\frac{4}{c^2}U+\frac{1}{c^2}a^2h|v|^2\right) \\ &\quad -\frac{4}{c}a\sqrt{h}\rho v^i,\ (i=1,2,3;\ j=4) \\ &= -\rho c v^j\left(1+\frac{\Pi}{c^2}+\frac{p}{\rho}+\frac{1}{c^2}a^2h|v|^2\right),(i=4;\ j=1,2,3) \\ &= -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{1}{c^2}a^2h|v|^2\right).(i=j=4). \end{aligned} \tag{10.3}$$

Now, we can receive U, V_i and S to post-Newtonian approximation in the homogeneous, isotropic universe similar to chapter 5.1. It follows

$$U = k\frac{\sqrt{h}}{a}\int\frac{\rho\prime}{|x-x\prime|}d^3x'. \tag{10.4}$$

Here, the integral is taken over the whole space and $\rho' = \rho(x', t)$, etc. We introduce

$$\chi = -k\frac{\sqrt{h}}{a}\int\rho'|x-x'|\,d^3x'. \tag{10.5}$$

We get

$$V_i = kh\int\rho'\frac{v^i\prime}{|x-x\prime|}d^3x' + \frac{1}{4}k\frac{dh}{dt}\int\rho'\frac{x^i-x^i\prime}{|x-x\prime|}d^3x'. \tag{10.6}$$

We introduce the potential

$$\psi = -kh\int\rho'\frac{(v',x-x')}{|x-x'|}d^3x' + \frac{1}{2}\frac{a}{\sqrt{h}}\frac{dh}{dt}\chi \tag{10.7a}$$

where $(v', x-x')$ denotes the scalar- product in R^3. It follows

$$S = 2U^2 + 2\frac{a}{\sqrt{h}}\psi + \left[16\pi k\left(\frac{\rho_{m0}}{a} + 2\frac{\rho_{r0}}{a^2}\right) - 2\Lambda c^2a^2\right]\chi$$
$$+a^2h\left\{\frac{d^2\chi}{dt^2} + \left(3\frac{a\prime}{a} + \frac{1}{2}\frac{h\prime}{h}\right)\frac{d\chi}{dt}\right\} - 4ah^{3/2}\phi_1 - 2\frac{\sqrt{h}}{a}\phi_3 - 6\frac{\sqrt{h}}{a}\phi_4 \tag{10.7b}$$

where ϕ_1, $\phi_3$and ϕ_4 are given by (5.14).

10.2 Equations of Motion

The conservation law of mass (1.27) implies by the use of

$$\rho^* = \rho\frac{dt}{d\tau} = \rho\sqrt{h}\left(1 + \frac{1}{c^2}U + \frac{1}{2c^2}a^2h|v|^2\right) \tag{10.8}$$

to $O\left(\frac{1}{c^2}\right)$ the well-known conservation law

$$\frac{\partial\rho^*}{\partial t} + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\rho^*v^k\right) = 0. \tag{10.9}$$

Hence, the conserved mass m is

$$m = \int\rho^*(x',t)\,d^3x'. \tag{10.10}$$

The equations of motion (5.35) use the Christoffel symbols $\Gamma(g)^i_{jk}$ which are omitted and can be found in the appendix of [*Pet* 00]. It is worth to mention that we have no gauge problem in contrast to the theory of general relativity considered by Shibata et al. [*Shi* 95] which implies some difficulties. To get the equations of motion given by (5.35) to post-Newtonian accuracy the energy tensor of matter $T(M)^{44}$ must be calculated to $O\left(\frac{1}{c^4}\right)$. Let us introduce the velocity

$$\bar{v}^i = a^2\sqrt{h}v^i\left(1 + \frac{1}{c^2}\left(\Pi + \frac{pc^2}{\rho} + U + \frac{1}{2}a^2h|v|^2\right)\right). \tag{10.11}$$

Replace U of (10.4) by

$$U^* = k\int \frac{\rho^{*\prime}}{|x-x'|}d^3x'. \tag{10.12}$$

Put

$$A(t) = \frac{a}{\sqrt{h}}\left\{h\left(-3\left(\frac{a'}{a}\right)^2 + \frac{a'}{a}\frac{h'}{h} + \frac{3}{4}\left(\frac{h'}{h}\right)^2\right) + H_0^2\left(\frac{21}{4}\frac{\Omega_m}{a^3} + \frac{23}{2}\frac{\Omega_r}{a^4} - \frac{21}{2}\Omega_\Lambda\right)\right\} \tag{10.13}$$

Then, the equations of motion to $O\left(\frac{1}{c^2}\right)$ are (i=1,2,3):

$$\begin{aligned}
&\frac{\partial\bar{v}^i}{\partial t} + \sum_{k=1}^3 v^k\frac{\partial\bar{v}^i}{\partial x^k} \\
&= -\frac{1}{\rho^*}\frac{\partial pc^2}{\partial x^i} + \frac{1}{a\sqrt{h}}\frac{\partial U^*}{\partial x^i} + \frac{1}{c^2}\frac{2}{a}U^*\frac{1}{\rho^*}\frac{\partial pc^2}{\partial x^i} \\
&\quad -\frac{7}{2}\frac{1}{c^2}\frac{a'}{a}\frac{k}{a}\int\rho^{*\prime}\frac{\bar{v}^{i\prime}}{|x-x'|}d^3x' \\
&\quad -\frac{1}{2}\frac{1}{c^2}\frac{k}{a}\int\rho^{*\prime}\frac{(\bar{v}',x-x')(x^i-x^{i\prime})}{|x-x'|^3}d^3x' \\
&\quad +\frac{1}{c^2}A(t)k\int\rho^{*\prime}\frac{x^i-x^{i\prime}}{|x-x'|}d^3x' \\
&\quad +\frac{1}{c^2}\frac{1}{a\sqrt{h}}\left(\Pi + 3\sqrt{h}\frac{pc^2}{\rho^*} + \frac{3}{2}\frac{|\bar{v}|^2}{a^2} - \frac{3}{a}U^*\right)\frac{\partial U^*}{\partial x^i} \\
&\quad -\frac{1}{c^2}\frac{k}{a\sqrt{h}}\int\rho^{*\prime}\Pi'\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'
\end{aligned} \tag{10.14}$$

$$-\frac{2}{c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}|\bar{v}'|^2\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{1}{c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}U^{*\prime}\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{3}{c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}\frac{(\bar{v}',x-x')\bar{v}^{i\prime}}{|x-x'|^3}d^3x'$$

$$+\frac{3}{2c^2}\frac{k}{a^3\sqrt{h}}\int \rho^{*\prime}\frac{(\tilde{v}',x-x')^2(x^i-x^{i\prime})}{|x-x'|^3}d^3x'$$

$$+\frac{7}{2c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}\frac{\partial U^{*\prime}}{\partial x^{i\prime}}\frac{1}{|x-x'|}d^3x'$$

$$+\frac{1}{2c^2}\frac{k}{a^2\sqrt{h}}\int \rho^{*\prime}\left(\sum_{k=1}^3(x^k-x^{k\prime})\frac{\partial U^{*\prime}}{\partial x^{k\prime}}\right)\frac{x^i-x^{i\prime}}{|x-x'|^3}d^3x'$$

$$-\frac{4}{c^2}\frac{k}{a}\int \rho^{*\prime}\left\{v^{i\prime}\frac{(v',x-x')}{|x-x'|^3}-\left(x^i-x^{i\prime}\right)\frac{|v'|^2}{|x-x'|^3}\right\}d^3x'$$

$$-\frac{2}{c^2}\frac{k}{a}v^i\int \rho^{*\prime}\frac{(v',x-x')}{|x-x'|^3}d^3x'-\frac{2}{c^2}a\sqrt{h}v^i\left(\sum_{k=1}^3 v^k\frac{\partial U^*}{\partial x^k}-\frac{a'}{a}U^*\right).$$

In the article [Shi 95] the special case $p=\Pi=0$ is considered. Furthermore, the two results of [Shi 95] and the equations of motion (10.14) with (10.8) cannot be compared with one another because the time-derivatives on the right side in [Shi 95] are not completely eliminated. In addition, the function $h(t)$ does not appear in Einstein's cosmological models. By the introduction of the proper time $\tilde{\tau}$ given by (8.1) the function $h(t)$ can be eliminated by

$$\tilde{v}^i=\frac{dv^i}{d\tilde{\tau}}=v^i\frac{dt}{d\tilde{\tau}}=v^i\sqrt{h},\ \frac{da}{d\tilde{\tau}}=\frac{da}{dt}\frac{dt}{d\tilde{\tau}}=a'\sqrt{h}. \tag{10.15}$$

By multiplication of relation (10.14) with $\sqrt{h}$ and the use of (10.15) it follows that h does not appear in the new equations of motion by the use of the proper time $\tilde{\tau}$ which is used by general relativity.

It is worth to mention that the special case where the universe is neglected, i.e.

$$\Omega_m=\Omega_r=\Omega_\Lambda=0, a(t)=h(t)=1,$$

the equations of motion (10.14) are studied in chapter 5.1 where the explicit form of the equations is not stated. In chapter V the energy-momentum (5.1) is used to compare post-Newtonian approximation of flat space-time of gravitation and of general relativity.

All the calculations of the represented results in this chapter can be found in the article [Pet 00].

10.3 Newtonian and Long-Field Forces

In this sub-chapter we will consider only the post-Newtonian long-field cosmological expression

$$F_i^U = \frac{1}{c^2} A(t) k \int \rho^{*\prime} \frac{x^i - x^{i\prime}}{|x - x\prime|} d^3 x' \tag{10.16}$$

of the non-stationary universe and compare (10.16) with the Newtonian force

$$F_i^N = -\frac{1}{a\sqrt{h}} k \int \rho^{*\prime} \frac{x^i - x^{i\prime}}{|x - x'|^3} d^3 x'. \tag{10.17}$$

In the following we consider spherical symmetry with Euclidean distance r from the centre of the body. We get

$$F_i^U = A(t) \frac{4\pi k}{c^2} \left(\int_0^r \rho^*(r') r'^2 \left(1 - \frac{1}{3} \left(\frac{r\prime}{r} \right)^2 \right) dr' + \frac{2}{3} r \int_r^\infty \rho^*(r') r' \, dr' \right) \frac{x^i}{r}$$

and

$$F_i^N = -\frac{1}{a\sqrt{h}} \frac{4\pi k}{r^2} \int_0^r \rho^* (r') {r'}^2 dr' \frac{x^i}{r}.$$

Let us assume that the radius r is greater than that of the distribution of matter and the post-Newtonian force expression (10.16) compensates the Newtonian force. We get compensation of the two forces for

$$\frac{1}{a\sqrt{h}} \frac{1}{r^2} \approx \frac{A(t)}{c^2}. \tag{10.18}$$

Hence, it follows

$$r \approx c / \left(a\sqrt{h} A(t) \right)^{1/2}. \tag{10.19}$$

Equation (9.3) implies that at present time $t = 0$, i.e. $a(0) = h(0) = 1$,and $A(0) \approx H_0^2$. Therefore, the post-Newtonian force $F_i^U$is only important on very large scales.

Let us consider the universe at later times, i.e. in the future. Then, it holds with

$$H_0\tilde{t}_1 \approx -1/\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)$$

for $H_0 t \gg H_0\tilde{t}_1$ by the results of chapter VII the relations

$$a^3(t) \approx \left(\frac{3}{2}\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0 t - H_0\tilde{t}_1)^2 \Big/ \left(3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}\right),$$

$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}.$$

Hence, for

$$H_0 t - H_0\tilde{t}_1 \gg \left(1-\sqrt{\Omega_m}\right)^2 \Big/ \left(3\Omega_m\sqrt{\Omega_\Lambda}\right)$$

we have

$$a^3(t) \approx \frac{3}{4}\frac{1}{\sqrt{\Omega_\Lambda}}\left(\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0 t - H_0\tilde{t}_1),$$
$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0 t - H_0\tilde{t}_1). \tag{10.20}$$

Elementary calculations yield

$$a\sqrt{h}A(t) \approx \frac{39}{2}\Omega_\Lambda H_0^2 a^2. \tag{10.21}$$

Hence, we get by the use of (10.19)

$$r \approx \sqrt{\frac{2}{39\Omega_\Lambda}}\left(\frac{4\sqrt{\Omega_\Lambda}\left(1-\sqrt{\Omega_\Lambda}\right)^2}{3\Omega_m{}^2}\right)^{1/3} \frac{c}{H_0}\frac{1}{(H_0 t - H_0\tilde{t}_1)^{1/3}}. \tag{10.22}$$

Therefore, the radius where the two forces compensate one another is decreasing with increasing time under the assumption $\Omega_\Lambda > 0$.

The case $\Omega_\Lambda = 0$ gives under the assumption (7.44b) by virtue of (7.45b) and (7.46) the solutions for the universe

$$a^3(t) \approx \frac{4}{9}\frac{1}{\Omega_m}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 (H_0 t - H_0\tilde{t}_1)^2,$$
$$\sqrt{h(t)} \approx 1 - \frac{1}{9}K_0,\ \ \Omega_m \approx 1. \tag{10.23}$$

Elementary calculations imply

$$a\sqrt{h}A(t) \approx \frac{9}{4}H_0^2\frac{1}{a}.$$

This result yields by the use of (10.19)

$$r \approx \frac{2}{3}\frac{c}{H_0}\sqrt{a(t)} \approx C_1\frac{c}{H_0}(H_0t - H_0\tilde{t}_1)^{1/3} \tag{10.24}$$

with a suitable constant C_1. Hence,the radius where the two forces compensate one another is increasing in the course of time.

Therefore, the radius where two forces compensate one another are quite different for the two cases. This radius decreases in a universe with $\Lambda > 0$ and increases in a universe with $\Lambda = 0$ in the course of time.

Summarizing, it follows that in the neighbourhood of a spherically symmetric body the large scale-structure in the universe is not important compared to the Newton force of this body. These results are also contained in the article [Pet 00].

Chapter 11

Preferred and Non-Preferred Reference Frames

11.1 Preferred Reference Frame

In this sub-chapter the preferred reference frame Σ' is shortly stated. In this frame Σ' the metric is the pseudo-Euclidean geometry, i.e.

$$(cd\tau')^2 = (ds')^2 = -\eta_{kl}{}'dx^{k\prime}dx^{l\prime} \tag{11.1a}$$

with

$$\eta_{11}{}' = \eta_{22}{}' = \eta_{33}{}' = 1,\, \eta'_{44} = -1,\ \eta'_{ij} = 0 \text{ (i}\neq\text{j).} \tag{11.1b}$$

In addition, the inverse tensor $\eta^{ij\prime}$ is given by

$$\eta'_{ik}\eta^{kj} = \delta_i^j. \tag{11.2a}$$

It follows

$$\eta^{11\prime} = \eta^{22\prime} = \eta^{33\prime} = 1, \eta^{44\prime} = -1, \eta^{ij\prime} = 0 \text{ (i}\neq\text{j).} \tag{11.2b}$$

Let $w' = \left(w^1, w^{2\prime}, w^{2\prime}\right)$ be a constant velocity vector and put

$$\gamma = \left(1 - \left|\frac{w\prime}{c}\right|^2\right)^{-1/2}. \tag{11.3}$$

Then, the Lorentz-transformations

$$\begin{aligned}\tilde{x}^{i\prime} &= x^{i\prime} + (\gamma - 1)\frac{(x', w\prime)}{|w\prime|^2}w^{i\prime} + \gamma t' w^{i\prime} \quad \text{(i=1, 2, 3)}\\ c\tilde{t}' &= \gamma\left(ct' + \left(x', \frac{w\prime}{c}\right)\right)\end{aligned} \tag{11.4}$$

do not change the line-element (11.1). All the quantities in Σ' are subsequently denoted with a prime and we put

$$x^{4\prime} = ct'. \tag{11.5}$$

The inverse formulae are

$$\begin{aligned}x^{i\prime} &= \tilde{x}^{i\prime} + (\gamma - 1)\frac{(\tilde{x}', w')}{|w'|^2}w^{i\prime} - \gamma\tilde{t}'w^{i\prime} \quad (i = 1, 2, 3)\\ x^{4\prime} &= \gamma\left(\tilde{x}^{4\prime} - \left(\tilde{x}', \frac{w'}{c}\right)\right).\end{aligned} \tag{11.6}$$

Hence, the transformations (11.4) and (11.6) give the possibility to transform a known event in Σ' into the same event moving with constant velocity w' in Σ'.

These are the well-known results of special relativity but the transformations (11.4) and (11.6) are always in in the same frame Σ' in contrast to the interpretation of special relativity where the transformations give the same result in a uniformly moving frame with velocity w'.The light velocity in the preferred frame Σ' is always the vacuum light velocity c.

11.2 Non-Preferred Reference Frame

Let us now consider a reference frame Σ which moves with velocity $-v' = -(v^{1\prime}, v^{2\prime}, v^{3\prime})$ relative to the preferred frame Σ'. All the results of this sub-chapter can be found in the article [Pet 86].

The non-preferred reference frame Σ is received from the preferred frame Σ' by the transformations

$$x^i = x^{i\prime} \text{ (i=1,2,3)}, \;\; x^4 = x^{4\prime} - \left(x', \frac{v\prime}{c}\right). \tag{11.7a}$$

The inverse transformation is

$$x^{i\prime} = x^i, \text{ (i=1,2,3)}, \; x^{4\prime} = x^4 + \left(x, \frac{v\prime}{c}\right). \tag{11.7b}$$

The metric follows from (11.1). We get

$$\begin{aligned} \eta_{ij} &= \delta_{ij} - \frac{v^{i\prime}}{c}\frac{v^{j\prime}}{c} \quad \text{(i; j= 1,2,3)} \\ &= -\frac{v^{i\prime}}{c} \quad \text{(1, 2, 3; j=4)} \\ &= -\frac{v^{j\prime}}{c} \quad \text{(i=1; j=1, 2, 3)} \\ &= -1. \qquad \text{(i=j=4)} \end{aligned} \tag{11.8a}$$

with

$$(cd\tau)^2 = -\eta_{kl} dx^k dx^l. \tag{11.8b}$$

The inverse has the form

$$\begin{aligned}\eta^{ij} &= \delta^{ij} \quad (i; j = 1, 2, 3)\\ &= -\frac{v^{i\prime}}{c} \quad (i = 1, 2, 3;\ j = 4)\\ &= -\frac{v^{j\prime}}{c} \quad (i = 1;\ j = 1, 2, 3)\\ &= -\left(1 - \left|\frac{v'}{c}\right|^2\right). \quad (i = j = 4)\end{aligned} \tag{11.9}$$

Elementary calculations give the absolute value of light-velocity

$$|v_l| = c/\left(1 - \left|\frac{v\prime}{c}\right| cos\vartheta\right) \tag{11.10}$$

where ϑ denotes the angle between the vectors $v_l$of light-velocity and v'. Hence the light-velocity is anisotropic.

We consider the Michelson-Morley experiment. Let l be the length of the arms of the apparatus. Then, the total time for the travelling of the ray is

$$t = \frac{l}{c}\left\{\left(1 - \left|\frac{v\prime}{c}\right| cos\vartheta\right) + \left(1 - \left|\frac{v'}{c}\right| cos(180^0 - \vartheta)\right)\right\} = \frac{2l}{c}. \tag{11.11}$$

Therefore, the null-result of Michelson-Morley is received. The transformations (11.7) give the result of an event studied in the preferred frame Σ' for the same event as it would appear in the non-preferred frame Σ and vice versa.

We will now study the transformations in Σ which correspond to the Lorentz-transformations in Σ', i.e. they transform an event in Σ as it appears in Σ when it has the velocity w' measured in Σ'. We have the formulae (11.7) and for the moving object the same transformations hold, i.e.

$$\tilde{x}^{i\prime} = \tilde{x}^i \quad \text{(i=1,2,3)}, \quad \tilde{x}^{4\prime} = \tilde{x}^4 + \left(\tilde{x}, \frac{v\prime}{c}\right). \tag{11.12}$$

The transformations (11.7) and (11.12) yield from the transformations (11.4) by elementary computations the result

$$\tilde{x}^i = x^i + (\gamma - 1)\frac{(x,w\prime)}{|w\prime|^2}w^{i\prime} + \gamma x^4\frac{w^i{}_\prime}{c} + \gamma\left(x,\frac{v\prime}{c}\right)\frac{w^i{}_\prime}{c}(1,2,3)$$

$$\tilde{x}^4 = \gamma\left(x^4 + \left(x,\frac{w'}{c}\right)\right) - \gamma\left(x^4 + \left(x,\frac{v'}{c}\right)\right)\left(\frac{w'}{c},\frac{v'}{c}\right) + (\gamma-1)\left(\left(x,\frac{v'}{c}\right) - \frac{\left(x,\frac{w'}{c}\right)}{|w'|^2}(w',v')\right). \tag{11.13a}$$

The inverse formulae are

$$x^i = \tilde{x}^i + (\gamma - 1)\frac{(\tilde{x},w\prime)}{|w\prime|^2}w^{i\prime} - \gamma\tilde{x}^4\frac{w^i{}_\prime}{c} - \gamma\left(\tilde{x},\frac{v\prime}{c}\right)\frac{w^i{}_\prime}{c} \quad (i=1,2,3)$$

$$x^4 = \gamma\left(\tilde{x}^4 - \left(\tilde{x},\frac{w'}{c}\right)\right) + \gamma\left(\tilde{x}^4 + \left(\tilde{x},\frac{v'}{c}\right)\right)\left(\frac{w'}{c},\frac{v'}{c}\right) + (\gamma-1)\left(\left(\tilde{x},\frac{v'}{c}\right) - \frac{\left(\tilde{x},\frac{w'}{c}\right)}{|w'|^2}(w',v')\right). \tag{11.13b}$$

Any event computed in Σ at rest can be calculated inΣ when it moves with velocity w'.

The four-velocity in Σ is

$$\left(\frac{dx^i}{d\tau}\right) = \frac{dt}{d\tau}\left(\frac{dx^1}{dt},\frac{dx^2}{dt},\frac{dx^3}{dt}\right)$$

and in Σ' the four-velocity is

$$\left(\frac{dx^i{}_\prime}{d\tau\prime}\right) = \frac{dt\prime}{d\tau\prime}\left(\frac{dx^{1\prime}}{dt\prime},\frac{dx^{2\prime}}{dt\prime},\frac{dx^{3\prime}}{dt\prime}\right).$$

The last two relations give by the use of (11.7) and the standard transformations for the velocities in Σ and Σ':

$$\frac{dx}{dt} = \frac{dx\prime}{dt\prime}\frac{1}{1-\left(\frac{1}{c}\frac{dx\prime}{dt'},\frac{v\prime}{c}\right)} \tag{11.14a}$$

$$\frac{dx'}{dt'} = \frac{dx}{dt}\frac{1}{1+\left(\frac{1}{c}\frac{dx}{dt},\frac{v'}{c}\right)}. \tag{11.14b}$$

In the special case that $\frac{dx^{i'}}{dt'} = v^{i'}$ we get

$$\frac{v}{c} = \frac{v'}{c}\frac{1}{1-\left|\frac{v'}{c}\right|^2}, \quad \frac{v'}{c} = \frac{v}{c}\frac{2}{1+\left(1+4\left|\frac{v}{c}\right|^2\right)^{1/2}}. \tag{11.15}$$

We will now give the transformation formulae for computing in the frame Σ an event which takes place in the frame Σ'. In the frame Σ the frame Σ' is described by the velocity $w' = v'$ in the formula (11.13), i.e.

$$\begin{aligned} \tilde{x}^i &= x^i + (\gamma-1)\frac{(x,v')}{|v'|^2}v^{i'} + \gamma x^4\frac{v^{i'}}{c} + \gamma\left(x,\frac{v'}{c}\right)\frac{v^{i'}}{c} \quad (i=1,2,3) \\ \tilde{x}^4 &= \gamma^{-1}\left(x^4 + \left(x,\frac{v'}{c}\right)\right). \end{aligned} \tag{11.16}$$

The transformation law from Σ to Σ' is given by (11.7a) which implies

$$\begin{aligned} \tilde{x}^i &= x^{i'} + (\gamma-1)\frac{(x',v')}{|v'|^2}v^{i'} + \gamma x^{4'}\frac{v^{i'}}{c} \quad (i=1,2,3) \\ \tilde{x}^4 &= \gamma^{-1}x^{4'}. \end{aligned} \tag{11.17a}$$

The inverse formulae are given by

$$\begin{aligned} x^{i'} &= \tilde{x}^i + (\gamma^{-1}-1)\frac{(\tilde{x},v')}{|v'|^2}v^{i'} - \gamma\tilde{x}^4\frac{v^{i'}}{c} \quad (i=1,2,3) \\ x^{4'} &= \gamma\tilde{x}^4. \end{aligned} \tag{1.17b}$$

The formulae (11.17) are given at first by Tangherlini [Tan 61] and later on by Marinov [Mar 80]. Marinov stated the measurement of the velocity of the Earth of about $\left|\frac{v'}{c}\right| \approx 10^{-3}$ in agreement with the observed velocity relative to the CMB. Hence, we can identify the Earth with the non-preferred frame Σ and the CMB frame with the preferred frame Σ'.

All these results can be found in the article of Petry [Pet 86]. Furthermore, the paper contains in the non-preferred frame Σ the equations of Maxwell in a medium, the equations of motion of a point particle in the electro-magnetic field.

In addition, the experiments of Hook and Fizeau are studied being in agreement with the observed results. The Doppler-effect is also studied in the reference frame Σ. All these studies are omitted here.

Chapter 12

Further Results

In this sub-chapter for some recently received experimental results theoretical essays are given to explain these results.

12.1 Anomalous Flyby

In this chapter an explanation of the anomalous Earth flyby is given. We follow along the lines of the article [Pet 11b]. Let us consider an observer in the preferred reference frame Σ' of the Earth. The boundary of the Earth is

$$X' = R\left(sin\left(\frac{2\pi}{T}t'\right)\cos\vartheta_0, -cos\left(\frac{2\pi}{T}t'\right)cos\vartheta_0, sin\vartheta_0\right) \tag{12.1}$$

where R denotes the radius of the Earth and T is the time of one day. The velocity of the boundary is given by

$$\frac{dX'}{dt'} = \frac{2\pi R}{T}\left(cos\left(\frac{2\pi}{T}t'\right)cos\vartheta_0, sin\left(\frac{2\pi}{T}t'\right)cos\vartheta_0, 0\right). \tag{12.2}$$

The velocity of a distant object (spacecraft) moving relative to the observer in Σ' can be given by

$$\frac{dx'}{dt'} = |w'|(cos\varphi cos\vartheta, sin\varphi cos\vartheta, sin\vartheta) \tag{12.3}$$

where φ and ϑ are fixed.

The motion of several space-crafts during the near Earth flyby shows an unexplained frequency shift which is interpreted as unexpected velocity change called Earth flyby anomaly. Let us now consider an observer on the boundary of the rotating Earth, i.e. in the non-preferred reference frame Σ moving with velocity

$$v' = \frac{dX'}{dt'}. \tag{12.4}$$

The proper- time in this frame is by the use of (11.8)

$$\begin{aligned}(cd\tau)^2 &= -|dx|^2 + \left(\frac{v'}{c}, dx\right)^2 + 2\left(\frac{v'}{c}, dx\right)dct + (dct)^2 \\ &= -|dx|^2 + \left(1 + \left(\frac{v'}{c}, \frac{1}{c}\frac{dx}{dt}\right)\right)^2 (dct)^2.\end{aligned} \tag{12.5}$$

Therefore, the transformations from the preferred reference frame Σ' of the non-rotating Earth into the preferred frame Σ of the rotating Earth can be given by (compare (11.7))

$$dx^{i\prime} = dx^i, \quad dx^{4\prime} = dx^4 \left(1 + \left(\frac{v\prime}{c}, \frac{1}{c}\frac{dx}{dt}\right)\right). \tag{12.6}$$

Let $k' = (k_1', k_2', k_3', k_4')$ be the wave four-vector of a plane wave in Σ' then the corresponding wave four-vector $k = (k_1, k_2, k_3, k_4)$ in Σ has by the transformation rules

$$k_i = k_l{}' \frac{\partial x^{l\prime}}{\partial x^i} \qquad \text{(i=1,2,3,4)}$$

the form

$$k_i = k_i{}' \quad \text{(i=1,2,3)}, \; k_4 = k_4{}' \left(1 + \left(\frac{v\prime}{c}, \frac{1}{c}\frac{dx}{dt}\right)\right).$$

The last relation gives for the frequency ν on the rotating Earth

$$\nu = \nu' \left(1 + \left(\frac{v\prime}{c}, \frac{1}{c}\frac{dx}{dt}\right)\right) \tag{12 7}$$

where ν' is the frequency measured in Σ'. In the frame Σ' the well-known Doppler-frequency formula

$$\nu' = \gamma \nu_0{}' \left(1 + \left|\frac{1}{c}\frac{dx\prime}{dt\prime}\right| cos\left(v_l'; \frac{dx\prime}{dt\prime}\right)\right)$$

holds where

$$\gamma = \left(1 - \left|\frac{1}{c}\frac{dx'}{dt'}\right|^2\right)^{-1/2}$$

and $\left(v_l'; \frac{dx\prime}{dt\prime}\right)$ denoted the angle between the light-velocity $v_l{}'$ and $\frac{dx\prime}{dt\prime}$. Therefore, we have in Σ by the use of (12.7) for the arriving frequency ν of the photon emitted by the moving object

$$\nu = \gamma \nu_0{}' \left(1 + \left|\frac{1}{c}\frac{dx\prime}{dt\prime}\right| cos\left(v_l'; \frac{dx\prime}{dt\prime}\right)\right) \left(1 + \left(\frac{v\prime}{c}, \frac{1}{c}\frac{dx}{dt}\right)\right).$$

This yields the frequency shift

$$\frac{\nu - \nu\prime}{\nu_0{}'} \approx \gamma \left(\frac{v\prime}{c}, \frac{1}{c}\frac{dx}{dt}\right). \tag{12.8}$$

Let the indices $*_a$ and $*_b$ mean after and before the perigee. Then (12.8) gives for the two-way frequency jump

$$2\frac{\Delta\nu}{\nu_0{'}} \approx 2\left\{\left(\frac{v_a{'}}{c}, \frac{1}{c}\frac{dx_a}{dt}\right) - \left(\frac{v_b{'}}{c}, \frac{1}{c}\frac{dx_b}{dt}\right)\right\} \tag{12.9}$$

Let us now assume that for distant objects

$$\left|\frac{dx_a}{dt}\right| \approx \left|\frac{dx_b}{dt}\right| \approx |w|.$$

We now apply the result (12.9) to the rotating Earth with velocity (12.4) with (12.2) and a distant object moving with velocity (12.3). It follows

$$2\frac{\Delta\nu}{\nu_0{'}} \approx \left|\frac{w}{c}\right|\frac{2\pi R}{T} cos\vartheta_0 \left\{cos\vartheta_a cos\left(\frac{2\pi}{T}t_a' - \varphi_a\right) - cos\vartheta_b cos\left(\frac{2\pi}{T}t_b{'} - \varphi_b\right)\right\} \tag{12.10}$$

where t' is the time when the photon emitted at the distant object arrives at the observer. For the special case

$$\frac{2\pi}{T}t_a' = \varphi_a, \ \frac{2\pi}{T}t_b' = \varphi_b \tag{12.11}$$

formula (12.10) gives the two-way frequency jump

$$2\frac{\Delta\nu}{\nu_0} \approx \left|\frac{w}{c}\right|\frac{2\pi R}{T}\cos\vartheta_0(cos\vartheta_a - cos\vartheta_b). \tag{12.12}$$

Hence, an observer at the poles, i.e. $\vartheta_0 = \pm\frac{\pi}{2}$ does not measure a frequency jump. Furthermore, there is no frequency jump when the spacecraft moves symmetrically about the plane of the equator, i.e. $\vartheta_a = -\vartheta_b$.

It is an open question whether the formula (12.12) or the more general formula (2.10) may explain the anomalous flyby of all the different spacecrafts.

It is worth to mention that the measured frequency jump doesn't imply a jump of the velocity of the spacecraft passing near the Earth. This may be the reason for the difficulty to explain the anomalous Earth flyby. The idea that the rotation of the Earth may explain the flyby anomaly is at first stated by Anderson [And 08]. It is worth to mention that the formula in [And 08] agrees with formula (12.12) for $\vartheta_0 = 0$, i.e. the observer is on the equator.

A great value of the flyby anomaly is measured by the spacecraft NEAR. The spacecraft ROSETTA showed in the years 2007 and 2009 no flyby anomaly (see e.g. http) whereas the formula in [And 08] predicts a small jump. It may be that formula (12.12) implies a very small frequency jump of ROSETTA if the observer has a suitable declination ϑ_0. Mbelek [Mbe 08] also studies the rotating Earth by the use of the transverse Doppler effect of special relativity and by some non-standard considerations.

General remarks about the problem of explaining the flyby anomaly can be found in [$http$].

12.2 Equations of Maxwell in a Medium

We consider a reference frame for which the pseudo-Euclidean geometry holds. The equations of Maxwell in empty space have a simple form and are derived from a Lagrangian. In a medium magnetic permeability and electric permittivity exist. The equations of Maxwell in a medium are also well-known but they cannot be derived as in empty space.

In addition to the pseudo-Euclidean metric a tensor of rank two is stated with which the proper-time in a medium is defined. The theory of Maxwell now follows along the lines of empty space. We follow the article of Petry [Pet $10b$]. Similar considerations can be found in the book of Hehl et al. [Heh 03] in the chapter about the metric by an alternative method.

We start from the pseudo-Euclidean metric (1.1), (1.4) and (1.5). The equations of Maxwell in a medium are well-known and are stated in many textbooks. They have the form

$$rot\ H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}J,\ div\ D = 4\pi\rho, \tag{12.13a}$$

$$rot\ E + \frac{1}{c}\frac{\partial B}{\partial t} = 0,\ div\ B = 0 \tag{12.13b}$$

with the electric current density $J = (J^1, J^2, J^3)$ and the electric charge ρ. We assume a simple medium with electric permittivity ε and magnetic permeability μ. The connection between electric and magnetic fields E and B and the derived fields D and B is given by

$$D = \varepsilon E,\ \ B = \mu H. \tag{12.14}$$

The absolute value of light-velocity in a medium is

$$|v_l| = c/\sqrt{\varepsilon\mu} = \frac{c}{n} \tag{12.15}$$

where n is the refraction index of the medium.

We now define in analogy to the theory of gravitation in flat space-time in addition to the metric the tensors

$$(g_{ij}) = \sqrt{\mu}diag\left(1,1,1,-\frac{1}{\varepsilon\mu}\right) \tag{12.16a}$$

with the inverse tensor

$$(g^{ij}) = \frac{1}{\sqrt{\mu}}diag(1,1,1,-\varepsilon\mu). \tag{12.16b}$$

The proper-time in the medium is given by

$$(cd\tau)^2 = -g_{kl}dx^k dx^l = -\sqrt{\mu}\left(|dx|^2 - \frac{1}{\epsilon\mu}(dct)^2\right). \tag{12.17}$$

We get from (12.17) by the use of $d\tau = 0$ the light-velocity (12.15).

Let A_i (i=1,2,3,4) be the electro-magnetic potentials and define the anti-symmetric tensors

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}. \tag{12.18a}$$

Furthermore, we define the tensors

$$F^{ij} = g^{ik}g^{jl}F_{kl} \tag{12.18b}$$

and let $J = (J^1, J^2, J^3, J^4)$ be the electric four-current density. We consider the covariant differential equations

$$\frac{\partial}{\partial x^k}F^{ki} = \frac{4\pi}{c}J^i \quad \text{(i=1,2,3,4)} \tag{12.19a}$$

$$\frac{\partial}{\partial x^k}F_{ij} + \frac{\partial}{\partial x^i}F_{jk} + \frac{\partial}{\partial x^j}F_{ki} = 0 \quad \text{(i,j,k=1,2,3,4)}. \tag{12.19b}$$

It immediately follows by the definition (12.18a) that the equations (12.19b) are fulfilled.

The electric field E and the magnetic field B are defined by

$$E = (F_{41}, F_{42}, F_{43}), \quad B = (F_{32}, F_{13}, F_{21}) \tag{12.20a}$$

then, the differential equations (12.19b) are identical with the equations (12.13b) of Maxwell.

Put for the derived fields

$$H = (F^{32}, F^{13}, F^{21}), \; D = (F^{14}, F^{24}, F^{34}). \qquad (12.20b)$$

Then, the differential equations (12.19a) are identical with the equations (12.13a) of Maxwell. It follows by (12.20) and (12,16)

$$H = \frac{1}{\mu} B, \quad D = \varepsilon E. \qquad (12.21)$$

Hence, we have received a reformulation of the equations of Maxwell in a medium similar to the equations of Maxwell in empty space.

Since the relations (12.19b) are fulfilled by the use of (12.18a) the potentials A_i must be calculated by relation (12.19a), i.e., for constant values of μ and ε

$$\frac{\partial}{\partial x^m} g^{mk} \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right) = \frac{4\pi}{c} g_{ik} J^k. \; \text{(i=1,2,3,4)}$$

By the use of the Lorentz-gauge

$$\frac{\partial}{\partial x^m} \left(g^{mk} A_k \right) = 0 \qquad (12.22)$$

the relation can be rewritten in the form

$$\frac{\partial}{\partial x^m} \left(g^{mk} \frac{\partial A_i}{\partial x^k} \right) = \frac{4\pi}{c} g_{ik} J^k. \quad \text{(i=1,2,3,4)} \qquad (12.23)$$

Hence, we get four differential equations (12.23) with the gauge condition (12.22) for the four potentials A_i (i=1,2,3,4).

The Lagrangian for the electro-magnetic field is

$$L_E = -\frac{1}{4} F^{kl} F_{kl} + \frac{4\pi}{c} A_k J^k \qquad (12.24)$$

with the definition (12.18). The energy-momentum tensor of the electro-magnetic field has the form

$$T(E)_j^i = \frac{1}{4\pi} \left(F^{ik} F_{jk} - \frac{1}{4} \delta_j^i F^{kl} F_{kl} \right). \qquad (12.25a)$$

The tensor

$$T(E)^{ij} = g^{jk} T(E)_k^i \qquad (12.25b)$$

is symmetric.

The equations of motion of a charged particle in the electro-magnetic field follow from the conservation law of the whole energy-momentum, i.e.

$$\frac{\partial}{\partial x^k}\left(T(E)_i^k + T(M)_i^k\right) = 0 \tag{12.26}$$

with

$$T(M)_j^i = \rho(E) g_{jk} \frac{dx^k}{d\tau}\frac{dx^i}{d\tau} \tag{12.27}$$

where $\rho(E)$ denotes the charge density.

All these results can be found in the article [$Pet\ 10b$]. The equations of Maxwell in a medium are also studied in a non-preferred reference frame (see sub-chapter 11.2).

12.3 Cosmological Models and the Equations of Maxwell in a Medium

We will now state a combination of electrodynamics in a medium (chapter 12.2) and the universe given by the use of absolute time t' by formula (8.6). The proper time in the universe is given by (8.6). In chapter 12.2 the used time t is the absolute time for the equations of Maxwell and in the universe the time t' is the absolute time. Therefore, it is ingenious to use in this sub-chapter for the equations of Maxwell in the universe the absolute time t'.

In the following we put as combination of chapter 12.2 and of the universe for the potentials of electrodynamics in a medium and of gravitation

$$\left(g_{ij}{}'\right) = a^2(t')\sqrt{\mu}\, diag\left(1,1,1,-\frac{1}{\varepsilon\mu}\right) \tag{12.28a}$$

with the inverse tensor

$$\left(g^{ij\prime}\right) = \frac{1}{a^2}\frac{1}{\sqrt{\mu}}\, diag(1,1,1,-\varepsilon\mu). \tag{12.28b}$$

Then, the proper-time τ has the form

$$(cd\tau)^2 = -a^2\sqrt{\mu}\left(|dx|^2 - \frac{1}{\varepsilon\mu}(dct')^2\right). \tag{12.29}$$

The absolute value of light-velocity is again stated by (12.15). The metric is by the use of (8.5):

$$(ds)^2 = -(|dx|^2 - (a^2h)(dct')^2). \tag{12.30}$$

In the following, the covariant derivatives relative to the metric (12.30) are used.

Define

$$G' = det(g_{ij}{}'), \quad \eta' = det(\eta_{ij}{}') \tag{12.31a}$$

and use the tensor (g_{ij}) given by (12.16). Put

$$G = det(g_{ij}), \quad \eta = det(\eta_{ij}). \tag{12.31b}$$

Furthermore, we use the pseudo-Euclidean metric (1.1) with (1.5).

Let A_i be the electro-magnetic potentials and define by the use of the covariant derivatives relative to the metric (12.30) the electro-magnetic field strength by

$$F_{ij} = A_{j/i} - A_{i/j}. \tag{12.32a}$$

It follows

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}. \tag{12.32b}$$

In addition, to the relations (12.32) we define a tensor $F^{ij}{}'$ (see chapter 12.2):

$$F^{ij'} = g^{ik'}g^{jl'}F_{kl}.$$

The covariant equations of Maxwell are given by

$$\left(\left(\frac{-G'}{-\eta'}\right)^{1/2} F^{ki'}\right)_{/k} = \frac{4\pi}{c}\left(\frac{-G}{-\eta}\right)^{1/2} J^i. \quad \text{(i=1,2,3,4)} \tag{12.33a}$$

In addition we have the covariant equations

$$F_{ij/k} + F_{jk/i} + F_{ki/j} = 0. \tag{12.33b}$$

The equation (12.33b) is identically fulfilled by virtue of (12.32a). We get from (12.28) and (12.16)

$$\left(g^{ij\prime}\right)=\frac{1}{a^2}\left(g^{ij}\right),\quad G'=-a^8\frac{\mu^2}{n^2},\ G=-\frac{\mu^2}{n^2},\ \eta'=-a^2h,\ \eta=-1.$$

The equations of Maxwell (12.33a) can be rewritten

$$\left(\frac{\mu}{n}\frac{1}{a\sqrt{h}}g^{km}g^{in}F_{mn}\right)_{/k}=\frac{4\pi}{c}\frac{\mu}{n}J^i.\quad \text{(i=1,2,3,4)}$$

We define for i,j=1,2,3,4 the tensor (in analogy to chapter 12.2):

$$F^{ij}=g^{ik}g^{jl}F_{kl}. \tag{12.34}$$

Then, the equations of Maxwell have the form

$$\left(\frac{\mu}{n}\frac{1}{a\sqrt{h}}F^{ki}\right)_{/k}=\frac{4\pi}{c}\frac{\mu}{n}J^i.\quad \text{(i=1,2,3,4)} \tag{12.35}$$

The only non-vanishing Christoffel symbol of the metric (12.30) is

$$\Gamma^4_{44}=\frac{1}{a\sqrt{h}}\frac{d}{dx^{4\prime}}\left(a\sqrt{h}\right).$$

Therefore, the equations of Maxwell are

$$\frac{1}{a\sqrt{h}}\frac{\partial}{\partial x^k}\left(\frac{\mu}{n}g^{km}g^{in}F_{mn}\right)=\frac{4\pi}{c}\frac{\mu}{n}J^i.\quad \text{(i=1,2,3,4)} \tag{12.36}$$

The definitions (12.18) and (12.20) give again the relations (12.21). Furthermore, the equation of Maxwell (12.36) has for constant $\mu\ and\ n$ the form

$$rot\ H-\frac{1}{c}\frac{\partial D}{\partial t'}=\frac{4\pi}{c}\left(a\sqrt{h}\vec{J}\right),\quad div\ D=\frac{4\pi}{c}\left(a\sqrt{h}\,J^4\right) \tag{12.37a}$$

where $\vec{J}=(J^1,J^2,J^3)$ and $\rho(E)=a\sqrt{h}\,J^4$.

The relations (12.33b) are rewritten in the form

$$rot\ E+\frac{1}{c}\frac{\partial B}{\partial t\prime}=0,\quad div\ B=0. \tag{12.37b}$$

The conservation of the streaming vector

$$J^k{}_{/k}=0$$

has the form

$$\frac{\partial}{\partial x^k}\left(a\sqrt{h}\,J^k\right)=0. \tag{12.38}$$

The equations (12.37) and (12.38) are the equations of Maxwell in a medium where (12.21) holds and which is contained in the universe. They are given relative to the metric (12.30).

12.4 Redshift of Distant Objects in a Medium

Here, we follow along the lines of article $[Pet\,13a]$. We assume that the proper-time τ is given by (12.29) where t' is the absolute time in the universe. We get by the use of (12.29) for an atom at rest which emits a photon at time $t_e{}'$

$$d\tau = a(t_e{}')\frac{\mu_e^{1/4}}{n_e}dt'. \tag{12.39}$$

Here, n_e and μ_e mean the refraction index and the permittivity of the medium in which the photon is emitted.

This means that the emitted frequency at time $t_e{}'$ is given by

$$\nu(t_e{}') = a(t_e{}')\frac{\mu_e^{1/4}}{n_e}\nu_0 \tag{12.40}$$

where ν_0 is the frequency emitted by the same atom at rest, at present time t_0' and without medium. The photon moves to the observer. The equations of motion (1.30) imply for i=4

$$\frac{d}{dt'}\left(g_{4k}{}'\frac{dx^{k\prime}}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{kl}}{\partial ct'}\frac{dx^{k\prime}}{dt'}\frac{dx^{l\prime}}{dt'}\frac{dt'}{d\tau}$$

We assume that the refraction index and the permittivity are depending on space but not on the time. This is justified by the equations of Maxwell (12.36) with the notations (12.32), (12.34), (12.16) and (12.21). Hence, we have

$$\frac{d}{dt'}\left(g_{44}{}'\frac{dct'}{d\tau}\right) = a\frac{da}{dct'}\mu^{1/2}\left(\left|\frac{dx'}{dt'}\right|^2 - \left(\frac{c}{n}\right)^2\right)\frac{dt'}{d\tau}$$

It follows by the use of (12.29)

$$\frac{d\tau}{dt'} = a\mu^{1/4}\frac{1}{c}\left(\left(\frac{c}{n}\right)^2 - \left|\frac{dx'}{dt'}\right|^2\right)^{1/2}$$

We get by the substitution of this relation into the above equation

$$\frac{d}{ct'}\left(g_{44}{}'\frac{dct'}{d\tau}\right) = -\mu^{1/4}\frac{da}{dt'}\left(\left(\frac{c}{n}\right)^2 - \left|\frac{dx'}{dt'}\right|^2\right)^{1/2} = 0$$

for the velocity of light, i.e. the energy of the photon during its motion is conserved. This means that the frequency is not changed by the use of the law of Planck. Hence, the frequency ν which arrives at the observer is

$$\nu = \nu(t_e{}') = a(t_e{}')\frac{\mu_e^{1/4}}{n_e}\nu_0.$$

This yields the redshift

$$z = \frac{\nu_0}{\nu} - 1 = \frac{1}{a(t_e{}')}\frac{n_e}{\mu_e^{1/4}} - 1. \qquad (12.41)$$

Taylor expansion of $a(t)$ yields

$$a(t_e') = a(t_0{}') + \dot{a}(t_0{}')(t_e' - t_0{}') + \frac{1}{2}\ddot{a}(t_0{}')(t_e' - t_0{}')^2 + \cdots$$

It is assumed that the photon's path from source to receiver is only a small fractional part that is within the medium, i.e. by virtue of (8.14) it holds $t_e' - t_0' = -\frac{r}{c}$. This implies by the use of the initial conditions

$$a(t_e{}') = 1 - H_0\frac{r}{c} + \frac{1}{2}\frac{\ddot{a}(t_0{}')}{H_0^2}\left(H_0\frac{r}{c}\right)^2 + \cdots \qquad (12.42a)$$

Relation (8.15) yields by the use of (7.17), (7.28) and $\Omega_r \ll 1$:

$$\frac{\ddot{a}(t_0{}')}{H_0^2} \approx 2 - \frac{3}{2}\Omega_m. \qquad (12.42b)$$

The redshift (12.41) is by the use of (12.42) given in the form

$$z = \frac{n_e}{\mu_e^{1/4}} - 1 + \frac{n_e}{\mu_e^{1/4}}\left(H_0\frac{r}{c}\right) + \frac{3}{4}\Omega_m\frac{n_e}{\mu_e^{1/4}}\left(H_0\frac{r}{c}\right)^2. \qquad (12.43)$$

The redshift formula (12.43) gives the whole value of the redshift. It follows partly from the universe and partly from the medium in which light is emitted. An intrinsic redshift is discussed by several authors who neglect an expanding universe. (see e.g. $[Fah\ 95]$). It is shown in the articles $[Pet\ 97], [Pet\ 07], [Pet\ 11a], [Pet\ 13b]$ that the redshift in the universe can also be interpreted with the aid of the different kinds of energy which are transformed into one another in the course of time as stated in chapter VIII. The interpretation of an expanding space is not necessary. An extensive study of a non-expanding universe exists, too (see e.g. $[Fah\ 95]$, $[Ler\ 05\ ]$,$[Alf\ 10]$).

Let us assume for the refraction index n and for the permittivity the representation

$$n_e = 1 + \Delta \mathrm{n}, \quad \mu_e = 1 + \Delta\mu.$$

Then, we get from (12.43)

$$z = \Delta n - \frac{1}{4}\Delta\mu + \frac{1}{4}(\Delta\mu)^2 + \frac{n_e}{\mu_e^{1/4}}\left(H_0 \frac{r}{c}\right) + \frac{3}{4}\Omega_m \frac{n_e}{\mu_e^{1/4}}\left(H_0 \frac{r}{c}\right)^2 + \cdots \quad (12.44)$$

Discussion:

(1) Relation (12.43) or (12.44) implies for a fixed redshift of a galaxy (quasar) in a medium that the distance to this object is in general, i.e.

$$\frac{n_e}{\mu_e^{1/4}} > 1,$$

smaller than without medium.

(2) The linear Hubble law can give an overestimate of the Hubble constant which depends also on the different media.

(3) Quasars may be nearer by the use of (12.43) or (12.44) than by the standard Hubble law. This yields that the measured energy emitted from these quasars is smaller than generally assumed.

(4) Two galaxies (quasars) in different media can give the same redshifts although the distances to these objects are different.

(5) Galaxies and quasars with nearly the same distances can have quite different redshifts which depend on the media in which light is emitted. Measurements confirm this result (see e.g., $[Arp\ 88], [Fah\ 95]$).

(6) It may be that dark energy is not necessary, i.e. $\Omega_\Lambda = 0$ because formula (12.43) or (12.44) may explain the redshifts of galaxies, of quasars, too.

Furthermore, there exists no age problem for the universe because the absolute timet' must be used instead of the proper-time $\tilde{\tau}$ (see (8.19)).

12.5 Flat Rotation Curves in Galaxies with Media

In this chapter we assume that every object, e.g. Earth, Sun, galaxy, quasar, etc. is surrounded by a medium. Furthermore, let us omit the universe, i.e. the objects are not too far from us. In addition to the object we consider the surrounding medium. In the following, only the approximations of Newton are used. Hence, we have $a(t) = h(t) = 1$ implying by the use of the Newtonian approximation the line-element of the pseudo-Euclidean metric and the proper-time

$$(cd\tau)^2 = -\left(|dx|^2 - \left(\frac{1}{n^2} - \frac{2}{c^2}U\right)(dct)^2\right). \tag{12.45}$$

Here, the Newtonian potential is given by (2.32) with (2.31) in the exterior of the object:

$$U = \frac{kM_g}{r}. \tag{12.46a}$$

Furthermore, let us assume that the refraction index is of the form

$$n = 1 + \Delta n \tag{12.46b}$$

with

$$0 < \Delta n \ll 1. \tag{12.46c}$$

This yields the approximate proper-time

$$(cd\tau)^2 = -\left(|dx|^2 - \left(1 - 2\Delta n - \frac{2}{c^2}U\right)(dct)^2\right). \tag{12.47}$$

The equations of motion (1.30) give to the lowest order

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^i}c^2 \quad \text{(i=1,2,3)}$$

i.e. we get

$$\frac{d^2x^i}{dt^2} = \frac{\partial \Delta n c^2}{\partial x^i} + \frac{\partial U}{\partial x^i} \quad \text{(i=1,2,3).} \tag{12.48}$$

Standard methods yield the result

$$\frac{1}{2}\frac{d}{dt}\left|\frac{dx}{dt}\right|^2 = \frac{d}{dt}(\Delta n c^2 + U). \tag{12.49}$$

Furthermore, we get an anomalous acceleration into the radial direction

$$b(r) = \frac{\partial \Delta n c^2}{\partial r}. \tag{12.50a}$$

We will now assume the simple form

$$\Delta n = b_0\left(1 - \frac{3}{2}\frac{r}{r_0} + \frac{1}{2}\left(\frac{r}{r_0}\right)^2\right) \tag{12.50b}$$

with $0 < b_0 \ll 1$ and where r_0 is the boundary of the medium. It is assumed that the boundary of the medium r_0 is great compared to the boundary of the body. A solution of (12.49) is given by

$$\left|\frac{dx}{dt}\right|^2 = 2\Delta n c^2 + 2U \tag{12.51}$$

where the constant of integration is set equal to zero. This result gives the rotation curves, as e.g. of galaxies, of stars etc. The derivation of this result doesn't correspond to the usual one of rotation curves but it has regard to the refraction index.

The equations (12.51), (12.50) and (12.46) give the rotation curves and the anomalous acceleration

$$\left|\frac{dx}{dt}\right| = \left(2b_0c^2\left(1 - \frac{3}{2}\frac{r}{r_0} + \frac{1}{2}\left(\frac{r}{r_0}\right)^2\right) + 2U\right)^{1/2} \tag{12.52a}$$

and

$$b(r) = -\frac{b_0c^2}{r_0}\left(\frac{3}{2} - \frac{r}{r_0}\right). \tag{12.52b}$$

It is worth to mention that for $r \ll r_0$ equation (12.52a) gives the well-known flat rotation curves.

The results (12.52) are applied to the Sun system and to galaxies:

(1) Sun system: We consider the Pioneers which give an anomalous acceleration (see e.g. [And 02]):

$$a_p \approx 8.74 \cdot 10^{-8}\,\frac{cm}{s^2} \tag{12.53}$$

into the direction to the Sun. There is an extensive study of the anomalous acceleration which is confirmed by several authors. Recently, Turyshev et al. [Tur 11] measured a decrease of the anomalous acceleration. This supports the explanation of an anisotropic emission of on-board heat which is also discussed by [And 02] and by many otherauthors.

Relation (12.52b) gives an anomalous acceleration to the Sun which implies by the use of (12.53) with $r \approx \frac{1}{2}r_0$ (r_0 in cm):

$$b_0 \approx 8.74 \cdot 10^{-8}\,\frac{r_0}{c^2} \approx 0.9 \cdot 10^{-28} r_0. \tag{12.54}$$

The result (12.52b) with (12.54) gives an anomalous acceleration which also decreases with the distance from the centre of the Sun. Hence, we have a quite different interpretation of the Pioneer anomaly without an anisotropic emission of on-board heat although the anisotropic emission is the presently accepted interpretation.

There are no flat rotation curves for the planets moving around the Sun by a suitable boundary r_0 of the medium. This follows by formula (12.52a) with (12.54).

(2) Galaxies:Many galaxies show flat rotation curves (see e.g. [San 86]). This result was already observed by Zwicky. Many authors assume dark matter to explain this result. But there are also other alternatives to explain the flat rotation curves. Milgrom [Mil 83] suggests a modified Newton law.

In this chapter the rotation curves are given by the use of (12.52a), i.e.

$$\left|\frac{dx}{dt}\right| = \left(2b_0c^2\left(1 - \frac{3}{2}\frac{r}{r_0} + \frac{1}{2}\left(\frac{r}{r_0}\right)^2\right) + 2U\right)^{1/2}. \tag{12.55}$$

Here, the last expression under the square root yields the well-known rotation curves. Let us assume that r_0 is very large then the condition

$$b_0c^2 > U = \frac{kM_g}{r} \tag{12.56}$$

gives flat rotation curves. The luminal mass of many galaxies is

$$M \approx \alpha 10^{10} M_{\odot} \approx 2\alpha 10^{43}\ g \tag{12.57}$$

with a suitable constant α depending on the galaxy.

The observed distance r where flat rotation curves arise is given by

$$r \geq \tilde{r} \cdot 10^{22}\ cm \tag{12.58}$$

with a suitable constant $\tilde{r}$. Relation (12.56) yields for flat rotation curves the condition

$$b_0 \geq 1.5\frac{\alpha}{\tilde{r}}10^{-7}. \tag{12.59}$$

This implies by the use of (12.52a) flat rotation curves with velocity

$$|v| \geq c \cdot 10^{-3}\sqrt{30\frac{\alpha}{\tilde{r}}}. \tag{12.60}$$

The inequality (12.60) gives the correct order of the velocity of flat rotation curves of galaxies. But every galaxy must be studied separately in detail.

Hence, the results about flat rotation curves of galaxies imply by the use of surrounding media the correct order of the measured velocities. Surrounding media of galaxies may therefore explain flat rotation curves without the assumption of dark matter. Contrary, all the dark matter contained in galaxies is not enough to explain all the dark matter in the universe. Therefore, media give the possibility to explain the results without the assumption of dark matter. Hence, we may ask whether media can be interpreted as the assumed dark matter.

Let us compute the density of dark matter produced by the reflection index. The law of Newton

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Delta n c^2}{dr}\right) = -4\pi k \rho_d \tag{12.61}$$

where ρ_d denotes the density of the assumed dark matter implied by the refraction index (12.50b). We get from formula (12.61):

$$\rho_d = 3\frac{bc^2}{4\pi k}\frac{1}{r_0}\frac{1}{r}\left(1-\frac{r}{r_0}\right). \tag{12.62}$$

The boundary of the dark matter is the radius r_0 given by (12.50b).The mass of the dark matter is by the use of (12.62)

$$M_d = \frac{4\pi}{3}\int r^2\rho_d\, dr = \frac{b_0 c^2}{6k} r_0. \tag{12.63}$$

Hence, it follows by the use of (12.54) that the assumed dark mass of the surrounding Sun is small compared to the mass of the Sun. But the dark mass of surrounding galaxies is for sufficiently large radius r_0 much greater than the luminous mass of the galaxy by virtue of (12.57), (12.58) and (12.59).

The density (12.62) of the assumed dark matter would give a singularity in the centre of the body but it is worth to mention that the medium surrounds the body and it is not the assumed dark matter.

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