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Zweier I - Convergent Sequence Spaces and Their Properties

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Preface

Sequence spaces play an important role in various fields of Real Analysis, Complex Analysis, Functional Analysis and Topology. These are very useful tools in demonstrating abstract concepts through constructing examples and counter examples. The topic "Sequence Spaces" is very broad in its own sense as one can study from various point of views, e.g. Schauder decomposition, $\alpha-$, $\beta-$, and $\gamma-$ duals, matrix transformations, measures of noncompactness, topological properties and geometric properties. The central theme of the present book is to introduce and study Zweier I-Convergent sequence spaces.

The structure of this text is straightforward. There are six chapters devoted to the various aspects of the theory. Each chapter is divided into sections. The numbers in the square brackets refers to the references listed in the bibliography.

As usual chapter 1, is devoted to the background materials which begins with the notations and conventions and some basic definitions which are needed throughout the work. This chapter concludes with an introduction to the Ideals which also includes some elementary properties.

In chapter 2, we introduce the Zweier I-convergent sequence spaces $\mathcal{Z}^I, \mathcal{Z}_0^I$ and $\mathcal{Z}_\infty^I$. We prove the decomposition theorem and study topological, algebraic properties and inclusion relations of these spaces.

In chapter 3, we introduce the Paranorm Zweier I-convergent sequence spaces $\mathcal{Z}^I(q), \mathcal{Z}_0^I(q)$ and $\mathcal{Z}_\infty^I(q)$ for $q = (q_k)$, a sequence of positive real numbers. We study some topological properties, prove the decomposition theorem and study some inclusion relations on these spaces.

In chapter 4, we introduce the sequence spaces $\mathcal{Z}^I(M), \mathcal{Z}_0^I(M)$ and $\mathcal{Z}_\infty^I(M)$ using the Orlicz function M. We study the algebraic properties and inclusion relations on these spaces.

In chapter 5, we introduce the sequence spaces $\mathcal{Z}^I(f)$, $\mathcal{Z}_0^I(f)$ and $\mathcal{Z}_\infty^I(f)$ for a modulus function f and study some of the topological and algebraic properties on these spaces.

In chapter 6, we introduce the sequence spaces $\mathcal{Z}^I(F)$, $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_\infty^I(F)$ for a sequence of modulii $F = (f_k)$ and study some of the topological and algebraic properties on these spaces.

In chapter 7, This is a precise chapter which is very special as it is designed only to study some inclusion relations between various zweier sequence spaces studied previously.

In chapter 8, we introduce the sequence spaces ${}_2\mathcal{Z}^I(F)$, ${}_2\mathcal{Z}_0^I(F)$ and ${}_2\mathcal{Z}_\infty^I(F)$ for a sequence of modulii $F = (f_k)$ and study some of the topological and algebraic properties on these spaces.

In chapter 9, we introduce the sequence spaces ${}_2\mathcal{Z}^I(f)$, ${}_2\mathcal{Z}_0^I(f)$ and ${}_2\mathcal{Z}_\infty^I(f)$ for a modulus function f and study some of the topological and algebraic properties on these spaces.

In chapter 10, we introduce the sequence spaces ${}_2\mathcal{Z}^I(M)$, ${}_2\mathcal{Z}_0^I(M)$, ${}_2\mathcal{Z}_\infty^I(M)$ for an Orlicz function M and study some of the topological and algebraic properties on these spaces.

The book ends with a fairly exhaustive bibliography of books and research articles consulted for the work.

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Contents

Chapter 1

Basic Definitions and Notations

“One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we originally put into them.”

The term sequence has a great role in Analysis. Convergence of sequences has always remained a subject of interest to the mathematicians. Several new types of convergence of sequences appeared, many of them are analogous to the statistical convergence. The concept of I-convergence gives a unifying approach to such type of convergence. Statistical convergence has several applications in different fields of Mathematics, Number Theory, Trigonometric Series, Summability Theory, Probability Theory, Measure Theory, Optimization and Approximation Theory. The notion of Ideal convergence corresponds to a generalization of the statistical convergence.

Notations

$\mathbb{N}$:= The set of all natural numbers.

$\mathbb{R}$:= The set of all real numbers.

$\mathbb{C}$:= The set of all complex numbers.

$\lim\limits_{k}$: means $\lim\limits_{k\to\infty}$.

$\sup\limits_{k}$: means $\sup\limits_{k\geq 1}$.

$\inf\limits_{k}$: means $\inf\limits_{k\geq 1}$, unless otherwise stated.

$\sum\limits_{k}$: means summation over $k = 1$ to $k = \infty$, unless otherwise stated.

$x := (x_k)$, the sequence whose k^{th} term is x_k.

$\theta := (0, 0, 0, ...)$, the zero sequence.

$e_k := (0, 0, .., 1, 0, 0, ..)$, the sequence whose k^{th} component is 1 and others are zeroes, for all $k \in \mathbb{N}$.

$e := (1, 1, 1, 1, ...)$.

$p := (p_k)$, the sequence of strictly positive reals.

$w := \{x = (x_k) : x_k \in \mathbb{R} \ \text{(or } \ \mathbb{C})\}$, the space of all sequences, real or complex.

$l : \{x \in w : \sum_k |x_k| < \infty\}$.

$l_\infty := \{x \in w : \sup_k |x_k| < \infty\}$, the space of bounded sequences.

$c_0 := \{x \in w : \lim_k |x_k| = 0\}$, the space of null sequences.

$c := \{x \in w : \lim_k x_k = l, \ \text{ for some } \ l \in \mathbb{C}\}$, the space of convergent sequences.

l_∞, c_0, c are Banach spaces with the usual norm

$$\|x\| = \sup_k |x_k|.$$

$l_1 := \{a = (a_k) : \sum_k |x_k| < \infty\}$, the space of absolutely convergent series.

$w_\infty := \{x \in w : \sup_n \frac{1}{n} \sum_k |x_k| < \infty\}$, the space of strongly Cesàro-bounded sequences.

$w_0 := \{x \in w : \lim_n \frac{1}{n} \sum_k |x_k| = 0\}$, the space of strongly Cesàro-null sequences.

$l_p := \{x \in w : \sum_k |x_k|^p < \infty\}, 0 < p < \infty$.

$w_p := \{x \in w : \lim_n \frac{1}{n} \sum_k |x_k - l|^p = 0; \ \text{ for some } \ l \in \mathbb{C}\}$.

In the case $1 \leq p < \infty$, the space l_p and w_p are Banach spaces normed by

$$\|x\| = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$$

and

$$\|x\| = \sup\left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}},$$

respectively. If $0 < p < 1$, then l_p and w_p are complete p-normed spaces, p-normed by

$$\|x\| = \sum_k |x_k|^p$$

and

$$\|x\| = \frac{1}{n}\sum_{k}^{n} |x_k|^p,$$

respectively.

The following subspaces of w were first introduced and discussed by Maddox [56] and Simons [69];

$$l(p) := \{x \in w : \sum_k |x_k|^{p_k} < \infty\}.$$

$$l_\infty(p) := \{x \in w : \sup_k |x_k|^{p_k} < \infty\}.$$

$$c(p) := \{x \in w : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\}.$$

$$c_0(p) := \{x \in w : \lim_k |x_k|^{p_k} = 0\}.$$

$$w_\infty(p) := \{x \in w : \sup_k (\tfrac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}) < \infty\}.$$

$$w(p) := \{x \in w : \lim_n (\tfrac{1}{n}\sum_{k=1}^{n} |x_k - l|^{p_k}) = 0, \text{ for some } l \in \mathbb{C}\}.$$

$$w_0(p) := \{x \in w : \lim_n (\frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k}) = 0\}.$$

Let $p = (p_k)$ be bounded. Then $c_0(p)$ is a linear metric space paranormed by:

$$g_1(x) = \sup_k |x_k|^{\frac{p_k}{M}},$$

where $M = \max(1, \sup_k p_k)$. $l_\infty(p)$ and $c(p)$ are paranormed by $g_1(x)$ defined above if and only if $\inf_k p_k > 0$. $l(p)$ and $w(p)$ are paranormed by:

$$g_2(x) = \left(\sum_k |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Remark 1.1. If $p_k = 1$, for all k, then $l_\infty(p) = l_\infty$, $c_0(p) = c_0$, $c(p) = c$, $l(p) = l$ and $w(p) = w$.

Definition 1.2. [48] A *paranorm* is a function $g : X \to \mathbb{R}$ which satisfies the following axioms: for any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$,

[i] $g(x) = 0$ if $x = \theta$;

[ii] $g(x) = g(-x)$;

[iii] $g(x + y) \leq g(x) + g(y)$;

[iv] the scalar multiplication is continuous, that is $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$. In other words,

$$|\lambda - \lambda_0| \to 0, \quad g(x - x_0) \to 0 \quad \text{imply} \quad g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and it is written as (X, g).

Any function g which satisfies all the conditions [i]-[iv] together with the condition

[v] $g(x) = 0$ if and only if $x = \theta$,

is called a *total paranorm* on X and the pair (X, g) is called *total paranormed space*.

Example 1.3. l_p is totally paranormed for any $p = (p_k) \in l_\infty$.

Definition 1.4. [68] Let X and Y be two nonempty subsets of the space w. Let $A = (a_{nk})$, $(n, k = 1, 2, ...)$ be an infinite matrix with elements of real or complex numbers.

We write

$$A_n(x) = \sum_k a_{nk} x_k,$$

provided the series converges. Then $Ax = (A_n(x))$ is called the *A-transform* of x.

Also

$$\lim_n Ax = \lim_{n\to\infty} A_n(x)$$

whenever it exists [68]. If $x \in X$ implies $Ax \in Y$, we say that A defines a (matrix) transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A that maps X into Y.

Definition 1.5. [58] A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$M\left(\frac{u+v}{2}\right) \le \frac{M(u) + M(v)}{2}, \qquad \text{for all } u, v \in \mathbb{R}.$$

If in addition, the two sides of above are not equal for $u \neq v$, then we call M to be strictly convex.

Definition 1.6. [55,58] A continuous function $M : \mathbb{R} \to \mathbb{R}$ is said to be *uniformly convex* if for any $\epsilon > 0$ and any $u_0 > 0$ there exists $\delta > 0$ such that

$$M\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{M(u)+M(v)}{2}, \qquad \text{for all } u, v \in \mathbb{R}$$

satisfying $|u-v| \geq \epsilon \max\{|u|, |v|\} \geq \epsilon u_0$.

Remark 1.7. If M is convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Definition 1.8. An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If convexity of M is replaced by $M(x+y) \leq M(x) + M(y)$, then it is called a *Modulus function*, defined and discussed by Nakano [58], Ruckle [62-64].

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Lindenstrauss and Tzafriri [55] used the idea of Orlicz function to construct the sequence space;

$$l_M := \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\};$$

which is a Banach space with the norm

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

Remark 1.9. An Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

For more details on Orlicz sequence spaces we refer to [55], [21-28].

Definition 1.10. An Orlicz function M is said to satisfy the $\varDelta_2 - condition$ $(M \in \varDelta_2$ for short$)$ if there exist constant $K \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

Definition 1.11. Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E, a sequence space, *The Multiplier Sequence* $E(\Lambda)$, associated with the sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. The concept of *Statistical convergence* was first introduced by Fast [12] and also independently by Schoenberg [67] for real and complex sequences.

Definition 1.12. [47] A sequence $x = (x_k)$ is called *Statistically Convergent* to L if

$$\lim_n \frac{1}{n}|\{k : |x_k - L| \geq \epsilon, k \leq n\}| = 0;$$

where the vertical bars indicate the number of elements in the set.

Remark 1.13. A sequence which converges statistically need not be convergent.

Example 1.14. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} k, & \text{if } k = n^2, \quad n \in N, \\ 0, & \text{otherwise ,} \end{cases}$$

and let $L = 0$. Then

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \subset \{1, 4, 9, 16, ..., k^2, ...\}.$$

We have that

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0, \text{ for every } \epsilon > 0.$$

This implies that the sequence (x_k) converges statistically to zero. But the sequence (x_k) does not converge to L.

Remark 1.15. A sequence which converges statistically need not be bounded. c.f([5], [7-9], [11], [13], [21-28], [29-38], [39-46].)

Asymptotic and Logarithmic Density 1.16. If $A \subseteq \mathbb{N}$, then χ_A denotes characteristic function of the set A, i.e.

$$\chi_A(k) = 1 \quad \text{if } k \in A$$

and

$$\chi_A(k) = 0 \quad \text{if } k \in \mathbb{N} - A.$$

Put

$$d_n(A) = \frac{1}{n} \sum_{k=1}^{n} \chi_A(k),$$

$$\delta_n(A) = \frac{1}{S_n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k},$$

where

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Then the numbers

$$\underline{d}(A) = \lim \inf_{n \to \infty} d_n(A),$$

$$\bar{d}(A) = \lim_{n\to\infty} \sup d_n(A),$$

are called the lower and upper asymptotic density of A, respectively(cf.[60],p.71). Similarly, the numbers

$$\underline{\delta}(A) = \lim_{n\to\infty} \inf \delta_n(A),$$

$$\bar{\delta}(A) = \lim_{n\to\infty} \sup \delta_n(A),$$

are called the lower and upper logarithmic density of A, respectively. If there exists

$$\lim_{n\to\infty} d_n(A) = d(A),$$

and

$$\lim_{n\to\infty} \delta_n(A) = \delta(A),$$

then $d(A)$ and $\delta(A)$ are called the asymptotic and logarithmic density of A respectively. It is well known fact, that for each $A \subseteq \mathbb{N}$,

$$\underline{d}(A) \leq \underline{\delta}(A) \leq \bar{\delta}(A) \leq \bar{d}(A).$$

Hence if $d(A)$ exists, then $\delta(A)$ also exists and $d(A) = \delta(A)$. The numbers $\underline{d}(A), \bar{d}(A), d(A), \underline{\delta}(A), \bar{\delta}(A), \delta(A)$ belong to the interval [0,1]. Owing to the well known formula

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = ln\, n + \gamma + 0(\frac{1}{n}), n \to \infty,$$

where γ is the Eulers constant.

Definition 1.17. Let $X \neq \phi$. A class $I \subset 2^X$ of subsets of X is said to be an *Ideal* in X if

[i] $\emptyset \in I$;

[ii] $A, B \in I$ imply $A \cup B \in I$;

[iii] $A \in I, B \subset A$ imply $B \in I$.

An ideal is called non-trivial if $X \notin I$ while an admissible ideal I further satisfies $\{x\} \in I$ for each $x \in X$.

Definition 1.18. Let $X \neq \phi$. A non-empty class $\pounds \subset 2^X$ of subsets of X is said to be *Filter* in X if

[i] $\emptyset \notin \pounds$;

[ii] $A, B \in \pounds$ imply $A \bigcap B \in \pounds$;

[iii] $A \in \pounds, B \supset A$ imply $B \in \pounds$,

The following Proposition expresses the relation between the notions of Ideals and Filters.

Proposition 1.19. Let I be a non-trivial ideal in X, $X \neq \phi$. Then the class

$$\pounds(I) = \{M \subseteq X : \exists A \in I : M = X - A\},$$

is a filter on X.

The concept of statistical convergence and the study of similar type of convergence lead to the introduction of the notion of I-convergence of sequences. The notion gives a unifying look at many types of convergence related to statistical convergence.

Definition 1.20. Let I be a non-trivial ideal in $\mathbb{N}$. A sequence $x = x_k$ of real numbers is said to be I-convergent to $\xi \in \mathbb{R}$ if for every $\epsilon > 0$ the set

$$A(\epsilon) = \{k : |x_k - \xi| \geq \epsilon\} \in I.$$

If $x = (x_k)$ is I-convergent to ξ we write $I - \lim x_k = \xi$ and the number ξ is called the I-limit of $x = (x_k)$.

The concept of I-convergence satisfies some usual axioms of convergence listed below:

[i] Every stationary sequence $x = (\xi, \xi, ...\xi, ...)$ I-converges to ξ.

[ii] The uniqueness of limit: If $I - \lim x = \xi$ and $I - \lim x = \eta$, then $\xi = \eta$.

[iii] If $I - \lim x = \xi$, then for each subsequence y of x we have $I - \lim y = \xi$.

[iv] If each subsequence y of a sequence x has a subsequence z I-convergent to ξ, then x is I-convergent to ξ.

Examples of Ideals 1.21.

[i] $I_0 = \emptyset$. This is the minimal non-empty non-trivial ideal in $\mathbb{N}$. A sequence is I_0 convergent if and only if it is constant.

[ii] Let $\phi \neq M \subseteq N$, $M \neq N$. Let$I_M = 2^M$. Then I_M is a non trivial ideal in N. A sequence $x = (x_k)$ is I_M-convergent if and only if it is constant on N-M.

[iii] Let I_f denotes the class of all finite subsets of $\mathbb{N}$. Then I_f is an admissible ideal in $\mathbb{N}$ and I_f-convergence coincides with the usual convergence in $\mathbb{R}$.

[iv] Let

$$I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}.$$

Then I_d is an admissible ideal in $\mathbb{N}$ and I_d-convergence coincides with the statistical convergence.

[v] Let

$$I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}.$$

Then I_δ is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence the logarithmic statistical convergence.

[vi] The examples [iv] and [v] can be generalised by choosing $c_n > 0$, such that

$$\sum_{n=1}^{\infty} c_n = +\infty.$$

Putting

$$h_m(A) = \frac{\sum\limits_{i\leq m, i\in A} c_i}{\sum\limits_{i=1}^{m} c_i} \qquad \text{(m=1,2,3......)} .$$

Denote by $h(A)$ the $\lim_{m\to\infty} h_m(A)$. Then

$$I_h = \{A \subseteq \mathbb{N} : h(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$ and I_d and I_δ-convergence are special cases of I_h convergence.

[vii] Let $u(A)$ denotes the uniform density of the set A. Then

$$I_u = \{A \subseteq \mathbb{N} : u(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$ and I_u-convergence will be called the uniform statistical convergence.

[viii] Let $T = (t_{n,k})$ be a non-negative regular matrix, then for each $A \subseteq \mathbb{N}$ the series

$$d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k}\chi_A(k) \qquad \text{(n=1,2,3......)} ,$$

converges if there exist

$$d_T(A) = \lim_{n\to\infty} d_T^{(n)}(A).$$

Then $d_T(A)$ is called the T-density of A. Putting

$$I_{d_T} = \{A \subseteq \mathbb{N} : d_T(A) = 0\},$$

then I_{d_T} is an admissible ideal in $\mathbb{N}$.

[ix] Let v be a finite additive measure defined on a class $\mathcal{U}$ of subsets of $\mathbb{N}$ which contains all finite subsets of $\mathbb{N}$ and $v(\{n\}) = 0$ for each $n \in \mathbb{N}$. $v(A) \leq v(B)$ if $A, B \in \mathcal{U}$, $A \subseteq B$. Then

$$I_v = \{A \subseteq \mathbb{N} : v(A) = 0\}$$

is an admissible ideal in $\mathbb{N}$.

[x] Let $\mu_m : 2^N \to [0, 1]$, m=1,2,.... be finitely additive measures defined on 2^N. If there exists

$$\mu(A) = \lim_{m\to\infty} \mu_m(A),$$

then $\mu(A)$ is called the measure of A, and

$$I_\mu = \{A \subseteq \mathbb{N} : \mu(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$.

[xi] Let

$$N = \bigcup_{j=1}^{\infty} D_j,$$

be a decomposition of N (i.e $D_k \cap D_l = \phi$ for $k \neq l$). Assume that $D_j(j = 1, 2, ...)$ are infinite sets. Choose $D_j = \{2^{j-1}(2s - 1) : s = 1, 2...\}$. Denote by $\mathcal{J}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of D_j. Then it is easy to see that $\mathcal{J}$ is an admissible ideal in $\mathbb{N}$.

[xii] The concept of density ρ of sets $A \subseteq \mathbb{N}$ is axiomatically introduced. Using this concept we can define the ideal

$$I_\rho = \{A \subseteq \mathbb{N} : \rho(A) = 0\},$$

and obtain I_ρ-convergence as a generalization of statistical convergence.

Relation between I-Convergence and μ-statistical Convergence 1.22.

The approach of Connor[7-9] towards the generalization of statistical convergence is based on using a finite additive measure μ defined on the field Γ of subsets of $\mathbb{N}$ with $\mu(\{k\}) = 0$ for each $k \in \mathbb{N}$ and such that $A, B \in \Gamma, A \subseteq B$ implies $\mu(A) \leq \mu(B)$. If we put

$$I = \{A \in \Gamma : \mu(A) = 0\},$$

then it is easy to verify that I is an admissible Ideal in $\mathbb{N}$ and

$$\pounds(I) = \{B \subseteq \mathbb{N} : \mu(B) = 1\}.$$

Conversely, if I is an admissible Ideal in $\mathbb{N}$, then we put

$$\Gamma = I \cup \pounds(I).$$

Then Γ is a field (Algebra) of subsets of $\mathbb{N}$. Define $\mu : \Gamma \to \{0, 1\}$ as follows:

$$\mu(M) = 0 \quad \text{if} \quad M \in I$$

$$\mu(M) = 1 \;\; \text{if} \;\; M \in \pounds(I).$$

Now it is easy to see that $I \cap \pounds(I) = \phi$ and $\mu(\{k\}) = 0$. Also the monotonicity and additivity of μ is preserved. Hence these two approaches towards generalization of statistical convergence seem to be equivalent in such a sense that each of them can be replaced by the other.

Fundamental arithmatical properties of I-convergence 1.23.

I-Convergence has arithmatical properties similar to the properties of the usual convergence.

Theorem 1.24. Let I be a non-trivial ideal in $\mathbb{N}$

(i) If $I-\lim x_n=\xi$, $I-\lim y_n=\eta$, then $I-\lim(x_n+y_n)=\xi+\eta$.

(ii) If $I-\lim x_n=\xi$, $I-\lim y_n=\eta$, then $I-\lim(x_n.y_n)=\xi.\eta$.

(iii) If I is an admissible ideal in $\mathbb{N}$, then $\lim_{n\to\infty} x_n=\xi$ implies $I-\lim x_n=\xi$.

Definition 1.25. A sequence $(x_k)\in\omega$ is said to be I-convergent to a number L if for every $\epsilon>0$. $\{k\in\mathbb{N}:|x_k-L|\geq\epsilon\}\in I$. In this case we write $I-\lim x_k=L$. The space c^I of all I-convergent sequences to L is given by

$$c^I=\{(x_k)\in\omega:\{k\in\mathbb{N}:|x_k-L|\geq\epsilon\}\in I, \text{for some L}\in\mathbb{C}\ \}.$$

Definition 1.26. A sequence $(x_k)\in\omega$ is said to be I-null if L = 0 . In this case we write $I-\lim x_k=0$.

Definition 1.27. A sequence $(x_k)\in\omega$ is said to be I-cauchy if for every $\epsilon>0$ there exists a number m = m(ϵ) such that $\{k\in\mathbb{N}:|x_k-x_m|\geq\epsilon\}\in$ I.

Definition 1.28. A sequence $(x_k)\in\omega$ is said to be I-bounded if there exists M >0 such that $\{k\in\mathbb{N}:|x_k|>M\}\in I$.

Definition 1.29. A map $\hbar$ defined on a domain $D\subset X$ i.e $\hbar:D\subset X\to\mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x)-\hbar(y)|\leq K|x-y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar\in(D,K)$.

Definition 1.30. A convergence field of I-convergence is a set

$$F(I)=\{x=(x_k)\in l_\infty:\text{there exists } I-\lim x\in\mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$.

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function.

Definition 1.31. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence α_k of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Definition 1.32. A sequence space E is said to be a sequence algebra if

$$(x_k) * (y_k) = (x_k y_k) \in E \quad \text{whenever} \quad (x_k), (y_k) \in E.$$

Definition 1.33. A sequence space E is said to be convergencefree if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.34. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$ where π is a permutation on $\mathbb{N}$.

Definition 1.35. A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces. (c.f.[2], [4], [6], [10], [17], [47], [48-49], [53-54], [60-61], [65-66], [70], [71-773], [74], [74], [76]).

Chapter 2

Zweier I-Convergent Sequence Spaces

“In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure.”- Hankel.

2.1 Introduction

Let l_∞, c and c_0 denote the Banach spaces of bounded,convergent and null sequences respectively normed by $||x||_\infty = \sup_k |x_k|$.

Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$ is called a sequence space.

A sequence space X with linear topology is called a K-space provided each of maps $p_i : X \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

A K-space λ is called an FK-space provided λ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers (a_{nk}), where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writing
$A : \lambda \longrightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{2.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of [2.1] converges for each $n \in \mathbb{N}$ and every $x \in \lambda$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay,Başar and Mursaleen[1], Başar and Altay[3], Malkowsky[57], Ng and Lee[59], and Wang[74]. Şengönül[68] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k), \\ 1-p, & (i-1 = k); (i, k \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Following Başar and Altay[3], Şengönül[68] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

Here we list below some of the results of [68] which we will need as a reference in order to establish analogously some of the results of this article.

Theorem 2.1.1. [68, Theorem 2.1] The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$||x||_{\mathcal{Z}} = ||x||_{\mathcal{Z}_0} = ||Z^p x||_c.$$

Theorem 2.1.2. [68, Theorem 2.2] The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$[See (Theorem 2.2.[18])]

Theorem 2.1.3. [68, Theorem 2.3] The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.

Theorem 2.1.4. [68, Theorem 2.6] $\mathcal{Z}_0$ is solid.

Theorem 2.1.5. [68, Theorem 3.6] $\mathcal{Z}$ is not a solid sequence space.

The following Lemmas will be used for establishing some results of this article.

Lemma 2.1.6. Let E be a sequence space. If E is solid then E is monotone. (see [20], page 53).

Lemma 2.1.7. If I $\subset 2^{\mathbb{N}}$ and M$\subseteq \mathbb{N}$. If M $\notin$I, then M$\cap\mathbb{N} \notin$I.
(see [71-72]).

2.2 Main Results

In this chapter we introduce the following classes of sequence spaces.

$$\mathcal{Z}^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = L, \text{ for some L} \in \mathbb{C}\} \in I\}$$

$$\mathcal{Z}_0^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = 0\} \in I\}$$

$$\mathcal{Z}_\infty^I = \{x = (x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

We also denote by

$$m_{\mathcal{Z}}^I = \mathcal{Z}_\infty \cap \mathcal{Z}^I$$

and

$$m_{\mathcal{Z}_0}^I = \mathcal{Z}_\infty \cap \mathcal{Z}_0^I$$

Throughout the article, for the sake of convenience now we will denote by $Z^p(x_k) = x^/, Z^p(y_k) = y^/, Z^p(z_k) = z^/$ for $x, y, z \in \omega$.

Theorem 2.2.1. The classes of sequences $\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I$ and let α, β be scalars. Then

$$I - \lim |x_k^{/} - L_1| = 0, \text{ for some } L_1 \in \mathbb{C};$$

$$I - \lim |y_k^{/} - L_2| = 0, \text{ for some } L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : |x_k^{/} - L_1| > \frac{\epsilon}{2}\} \in I, \qquad [2.2]$$

$$A_2 = \{k \in \mathbb{N} : |y_k^{/} - L_2| > \frac{\epsilon}{2}\} \in I. \qquad [2.3]$$

we have

$$\begin{aligned} |(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| &\leq |\alpha|(|x_k^{/} - L_1|) + |\beta|(|y_k^{/} - L_2|) \\ &\leq |x_k^{/} - L_1| + |y_k^{/} - L_2| \end{aligned}$$

Now, by [2.2] and [2.3], $\{k \in \mathbb{N} : |(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I$

Hence $\mathcal{Z}^I$ is a linear space.

Theorem 2.2.2. The spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are normed linear spaces,normed by

$$||x_k^{/}||_* = \sup_k |Z^p(x)|. \qquad [2.4]$$

where $x_k^{/} = Z^p(x)$

Proof. It is clear from Theorem 2.2.1 that $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces. It is easy to verify that [2.4] defines a norm on the spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 2.2.3. A sequence $x = (x_k) \in m_{\mathcal{Z}}^I$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I \qquad [2.5]$$

Proof. Suppose that $L = I - \lim x^/$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k^/ - L| < \frac{\epsilon}{2}\} \in m_{\mathcal{Z}}^I \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon}^/ - x_k^/| \leq |x_{N_\epsilon}^/ - L| + |L - x_k^/| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$.

Conversely, suppose that $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$. That is $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k^/ \in [x_{N_\epsilon}^/ - \epsilon, x_{N_\epsilon}^/ + \epsilon]\} \in m_{\mathcal{Z}}^I \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon}^/ - \epsilon, x_{N_\epsilon}^/ + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m_{\mathcal{Z}}^I$ as well as $C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k^/ \in J\} \in m_{\mathcal{Z}}^I$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \ldots\ldots \supseteq I_k \supseteq \ldots\ldots\ldots\ldots$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k^{/} \in I_k\} \in m_{\mathcal{Z}}^I$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi^{/} = I - \lim x^{/}$, that is $L = I - \lim x^{/}$.

Theorem 2.2.4. Let I be an admissible ideal. Then the following are equivalent.

(a) $(x_k) \in \mathcal{Z}^I$;

(b) there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I;

(c) there exists $(y_k) \in \mathcal{Z}$ and $(z_k) \in \mathcal{Z}_0^I$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L| \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

Proof. (a) implies (b). Let $(x_k) \in \mathcal{Z}^I$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : |x_k^{/} - L| \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : |x_k^{/} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_k) as

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x_k^{/} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k^{/} - L| \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c). For $(x_k) \in \mathcal{Z}^I$. Then there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $K \in I$. Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I$ and $y_k \in \mathcal{Z}$.

(c) implies (d). Let $P_1 = \{k \in \mathbb{N} : |z_k| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$. Then for any $\epsilon > 0$, and Lemma , we have

$$\{k \in \mathbb{N} : |x_k^{/} - L| \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x_k^{/} - L| \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I$.

Theorem 2.2.5. The inclusions $\mathcal{Z}_0^I \subset \mathcal{Z}^I \subset \mathcal{Z}_\infty^I$ are proper.

Proof. Let $(x_k) \in \mathcal{Z}^I$. Then there exists L $\in \mathbb{C}$ such that

$$I - \lim |x_k^{/} - L| = 0$$

We have $|x_k^{/}| \leq \frac{1}{2}|x_k^{/} - L| + \frac{1}{2}|L|$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I$. The inclusion $\mathcal{Z}_0^I \subset \mathcal{Z}^I$ is obvious.

Theorem 2.2.6. The function $\hbar : m^I_{\mathcal{Z}} \to \mathbb{R}$ is the Lipschitz function, where
$m^I_{\mathcal{Z}} = \mathcal{Z}^I \cap \mathcal{Z}_\infty$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x^/_k - \hbar(x^/)| \geq ||x^/ - y^/||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y^/_k - \hbar(y^/)| \geq ||x^/ - y^/||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x^/_k - \hbar(x^/)| < ||x^/ - y^/||_*\} \in m^I_{\mathcal{Z}},$$

$$B_y = \{k \in \mathbb{N} : |y^/_k - \hbar(y^/)| < ||x^/ - y^/||_*\} \in m^I_{\mathcal{Z}}.$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x^/) - \hbar(y^/)| \leq |\hbar(x^/) - x^/_k| + |x^/_k - y^/_k| + |y^/ - \hbar(y^/)| \leq 3||x^/ - y^/||_*.$$

Thus $\hbar$ is a Lipschitz function. For $m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.2.7. If $x, y \in m^I_{\mathcal{Z}}$, then $(x.y) \in m^I_{\mathcal{Z}}$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x^/ - \hbar(x^/)| < \epsilon\} \in m^I_{\mathcal{Z}},$$

$$B_y = \{k \in \mathbb{N} : |y^/ - \hbar(y^/)| < \epsilon\} \in m^I_{\mathcal{Z}}.$$

Now,

$$\begin{aligned} |x^/.y^/ - \hbar(x^/)\hbar(y^/)| &= |x^/.y^/ - x^/\hbar(y^/) + x^/\hbar(y^/) - \hbar(x^/)\hbar(y^/)| \\ &\leq |x^/||y^/ - \hbar(y^/)| + |\hbar(y^/)||x^/ - \hbar(x^/)| \qquad [2.6] \end{aligned}$$

As $m^I_{\mathcal{Z}} \subseteq \mathcal{Z}_\infty$, there exists an $M \in \mathbb{R}$ such that $|x^/| < M$ and $|\hbar(y^/)| < M$. Using eqn[2.6] we get

$$|x^/.y^/ - \hbar(x^/)\hbar(y^/)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m^I_{\mathcal{Z}}$. Hence $(x.y) \in m^I_{\mathcal{Z}}$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For $m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.2.8. The spaces $\mathcal{Z}^I_0$ and $m^I_{\mathcal{Z}_0}$ are solid and monotone .

Proof. We shall prove the result for $\mathcal{Z}^I_0$. Let $(x_k) \in \mathcal{Z}^I_0$. Then

$$I - \lim_k |x^/_k| = 0 \qquad [2.7]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [2.7] and the following inequality $|\alpha_k x^/_k| \leq |\alpha_k||x^/_k| \leq |x^/_k|$ for all $k \in \mathbb{N}$. That the space $\mathcal{Z}^I_0$ is monotone follows from the Lemma 2.1.6. For $m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.2.9. The spaces $\mathcal{Z}^I$ and $m^I_{\mathcal{Z}}$ are neither monotone nor solid, if I is neither maximal nor $I = I_f$ in general .

Proof. Here we give a counter example. Let I = I$_\delta$. Consider the K-step space X_K of X defined as follows, Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y^/_k) = \begin{cases} (x^/_k), & \text{if k is odd,} \\ 1, & otherwise. \end{cases}$$

Consider the sequence $(x^/_k)$ defined by $(x^/_k) = k^{-1}$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I$. Thus $\mathcal{Z}^I$ is not monotone. Hence $\mathcal{Z}^I$ is not solid.

Theorem 2.2.10. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}^I_0$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I$. Then

$$I - \lim |x_k^/| = 0$$

and

$$I - \lim |y_k^/| = 0$$

Then we have

$$I - \lim |(x_k^/ . y_k^/)| = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I$. Hence $\mathcal{Z}_0^I$ is a sequence algebra. For the space $\mathcal{Z}^I$, the result can be proved similarly.

Theorem 2.2.11. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$. Consider the sequence $(x_k^/)$ and $(y_k^/)$ defined by

$$x_k^/ = \frac{1}{k} \text{ and } y_k^/ = k \text{ for all } k \in \mathbb{N}$$

Then $(x_k) \in \mathcal{Z}^I$ and $\mathcal{Z}_0^I$, but $(y_k) \notin \mathcal{Z}^I$ and $\mathcal{Z}_0^I$. Hence the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free.

Theorem 2.2.12. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Proof. Let $A \in I$ be infinite. If

$$x_k^/ = \begin{cases} 1, & \text{for } k \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.16. $x_k \in \mathcal{Z}_0^I \subset \mathcal{Z}^I$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin \mathcal{Z}^I$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I$. Hence $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Theorem 2.2.13. The sequence spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are linearly isomorphic to the spaces c^I and c_0^I respectively, i.e $\mathcal{Z}^I \cong c^I$ and $\mathcal{Z}_0^I \cong c_0^I$.

Proof. We shall prove the result for the space $\mathcal{Z}^I$ and c^I. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces $\mathcal{Z}^I$ and c^I. Define a map $T : \mathcal{Z}^I \longrightarrow c^I$ such that $x \to x^{/} = Tx$

$$T(x_k) = px_k + (1-p)x_{k-1} = x_k^{/}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x_k^{/} \in c^I$ and define the sequence $x = x_k$ by

$$x_k = M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^{/}. \quad (i \in \mathbb{N})$$

where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{k\to\infty} px_k + (1-p)x_{k-1} = p \lim_{k\to\infty} M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^{/} +$$

$$(1-p) \lim_{k\to\infty} M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x_i^{/} = \lim_{k\to\infty} x_k^{/}$$

which shows that $x \in \mathcal{Z}^I$.

Hence T is a linear bijection. Also we have $||x||_* = ||Z^p x||_c$. Therefore

$$||x||_* = \sup_{k\in\mathbb{N}} |px_k + (1-p)x_{k-1}|$$

$$= \sup_{k\in\mathbb{N}} |pM\sum_{i=0}^{k}(-1)^{k-i}N^{k-i}x_i^{/} + (1-p)M\sum_{i=0}^{k-1}(-1)^{k-i}N^{k-i}x_i^{/}|$$

$$= \sup_{k\in\mathbb{N}} |x_k^{/}| = ||x^{/}||_{c^I}$$

Hence $\mathcal{Z}^I \cong c^I$.

Chapter 3

On Paranorm Zweier I-Convergent Sequence Spaces

"There is no place in the world for ugly mathematics. It may be very hard to define mathematical beauty but that is just as true of beauty of any kind , we may not know quite, what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it."-Hardy

3.1 Introduction

The following subspaces of ω were first introduced and discussed by Maddox [56] :

$l(p) := \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$,

$l_\infty(p) := \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$,

$c(p) := \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C} \}$,

$c_0(p) := \{x \in \omega : \lim_k |x_k|^{p_k} = 0, \}$,

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides[53-54] defined the following sequence spaces :

$l_\infty\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\}$,

$c_0\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0, \}$,

$l\{p\} := \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\}$,

Where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

Recently Khan and Ebadullah [38] introduced the following classes of sequence spaces:

$$\mathcal{Z}^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = L \text{ for some L}\} \in I\};$$

$$\mathcal{Z}_0^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = 0\} \in I\};$$

$$\mathcal{Z}_\infty^I = \{(x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

We also denote by

$$m_{\mathcal{Z}}^{I} = \mathcal{Z}_{\infty} \cap \mathcal{Z}^{I};$$

and

$$m_{\mathcal{Z}_0}^{I} = \mathcal{Z}_{\infty} \cap \mathcal{Z}_0^{I}.$$

In this chapter we introduce the following classes of sequence spaces:

$$\mathcal{Z}^{I}(q) = \{(x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x - L|^{q_k} \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C} \ \};$$

$$\mathcal{Z}_0^{I}(q) = \{(x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x|^{q_k} \geq \epsilon\} \in I\};$$

$$\mathcal{Z}_{\infty}^{I}(q) = \{(x_k) \in \omega : \sup_k |Z^p x|^{q_k} < \infty\}.$$

We also denote by

$$m_{\mathcal{Z}}^{I}(q) = \mathcal{Z}_{\infty}^{I}(q) \cap \mathcal{Z}^{I}(q);$$

and

$$m_{\mathcal{Z}_0}^{I}(q) = \mathcal{Z}_{\infty}^{I}(q) \cap \mathcal{Z}_0^{I}(q);$$

where $q = (q_k)$, is a sequence of positive real numbers.

Throughout the chapter, for the sake of convenience we will denote by $Z^p x = x^/, Z^p y = y^/, Z^p z = z^/$ for all $x, y, z \in \omega$.

3.2 Main Results

Theorem 3.2.1. The classes of sequences $\mathcal{Z}^{I}(q), \mathcal{Z}_0^{I}(q), m_{\mathcal{Z}}^{I}(q)$ and $m_{\mathcal{Z}_0}^{I}(q)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^{I}(q)$. The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I(q)$ and let α, β be scalars. Then for a given $\epsilon > 0$ we have

$$\{k \in \mathbb{N} : |x_k^/ - L_1|^{q_k} \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \ \} \in I;$$

$$\{k \in \mathbb{N} : |y_k^/ - L_2|^{q_k} \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \ \} \in I;$$

where

$$M_1 = Dmax\{1, \sup_k |\alpha|^{q_k}\};$$
$$M_2 = Dmax\{1, \sup_k |\beta|^{q_k}\};$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k q_k \geq 0.$$

Let

$$A_1 = \{k \in \mathbb{N} : |x_k^/ - L_1|^{q_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \ \} \in \pounds(I);$$

$$A_2 = \{k \in \mathbb{N} : |y_k^/ - L_2|^{q_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \ \} \in \pounds(I);$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 = \{k \in \mathbb{N} : |(\alpha x_k^/ + \beta y_k^/) - (\alpha L_1 + \beta L_2)|^{q_k}) < \epsilon\} \\ \supseteq \{k \in \mathbb{N} : |\alpha|^{q_k}|x_k^/ - L_1|^{q_k} < \frac{\epsilon}{2M_1}|\alpha|^{q_k}D\} \\ \cap \{k \in \mathbb{N} : |\beta|^{q_k}|y_k^/ - L_2|^{q_k} < \frac{\epsilon}{2M_2}|\beta|^{q_k}D\}. \end{aligned}$$

Thus $A_3^c \subseteq A_1^c \cup A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(q)$. Therefore $\mathcal{Z}^I(q)$ is a linear space. The rest of the result follows similarly.

Theorem 3.2.2. Let $(q_k) \in l_\infty$. Then $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are paranormed spaces, paranormed by

$$g(x) = \sup_k |x_k|^{\frac{q_k}{M}}, \text{ where } M = max\{1, \sup_k q_k\}.$$

Proof. Let $x = (x_k), y = (y_k) \in m^I_{\mathcal{Z}}(q)$.

[i] Clearly, $g(x) = 0$ if and only if $x = 0$.

[ii] $g(x) = g(-x)$ is obvious.

[iii] Since $\frac{q_k}{M} \leq 1$ and $M > 1$,using Minkowski's inequality we have

$$\sup_k |x_k + y_k|^{\frac{q_k}{M}} \leq \sup_k |x_k|^{\frac{q_k}{M}} + \sup_k |y_k|^{\frac{q_k}{M}}.$$

[iv] Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$. Let $x_k \in m^I_{\mathcal{Z}}(q)$ such that $|x_k - L|^{q_k} \geq \epsilon$. Therefore,

$$g(x - Le) = \sup_k |x_k - L|^{\frac{q_k}{M}} \leq \sup_k |x_k|^{\frac{q_k}{M}} + \sup_k |L|^{\frac{q_k}{M}},$$

where $e = (1, 1, 1.....)$. Hence

$$g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x) + \lambda g(L),$$

as $k \to \infty$. Hence $m^I_{\mathcal{Z}}(q)$ is a paranormed space. The rest of the result follows similarly.

Theorem 3.2.3. $m^I_{\mathcal{Z}}(q)$ is a closed subspace of $l_\infty(q)$.

Proof. Let $(x^{(n)}_k)$ be a Cauchy sequence in $m^I_{\mathcal{Z}}(q)$ such that $x^{(n)} \to x$. We show that $x \in m^I_{\mathcal{Z}}(q)$. Since $(x^{(n)}_k) \in m^I_{\mathcal{Z}}(q)$, then there exists a_n such that

$$\{k \in \mathbb{N} : |x^{(n)} - a_n| \geq \epsilon\} \in I.$$

We need to show that

[i] (a_n) converges to a.

[ii] If $U = \{k \in \mathbb{N} : |x_k - a| < \epsilon\}$, then $U^c \in I$.

[i] Since $(x_k^{(n)})$ is a Cauchy sequence in $m_{\mathcal{Z}}^I(q)$ then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}, \quad \text{for all n, i} \geq k_0$$

For a given $\epsilon > 0$, we have

$$B_{ni} = \{k \in \mathbb{N} : |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}\},$$

$$B_i = \{k \in \mathbb{N} : |x_k^{(i)} - a_i| < \frac{\epsilon}{3}\},$$

$$B_n = \{k \in \mathbb{N} : |x_k^{(n)} - a_n| < \frac{\epsilon}{3}\}.$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$.

Let

$$B^c = B_{ni}^c \cup B_i^c \cup B_n^c,$$

where

$$B = \{k \in \mathbb{N} : |a_i - a_n| < \epsilon\}.$$

Then $B^c \in I$. We choose $k_0 \in B^c$, then for each $n, i \geq k_0$, we have

$$\{k \in \mathbb{N} : |a_i - a_n| < \epsilon\} \supseteq \{k \in \mathbb{N} : |x_k^{(i)} - a_i| < \frac{\epsilon}{3}\}$$

$$\cap \{k \in \mathbb{N} : |x_k^{(n)} - x_k^{(i)}| < \frac{\epsilon}{3}\} \cap \{k \in \mathbb{N} : |x_k^{(n)} - a_n| < \frac{\epsilon}{3}\}.$$

Then (a_n) is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $a_n \to a$, as $n \to \infty$.

[ii] Let $0 < \delta < 1$ be given. Then we show that if

$$U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\},$$

then $U^c \in I$. Since $x^{(n)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : |x^{(q_0)} - x| < (\frac{\delta}{3D})^M\}. \qquad [3.1]$$

which implies that $P^c \in I$.

The number q_0 can be so chosen that together with [3.1], we have

$$Q = \{k \in \mathbb{N} : |a_{q_0} - a|^{q_k} < (\frac{\delta}{3D})^M\},$$

such that $Q^c \in I$

Since

$$\{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} \geq \delta\} \in I.$$

Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} < (\frac{\delta}{3D})^M\}.$$

Let

$$U^c = P^c \cup Q^c \cup S^c,$$

where

$$U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\}.$$

Therefore for each $k \in U^c$, we have

$$\{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\} \supseteq \{k \in \mathbb{N} : |x^{(q_0)} - x|^{q_k} < (\frac{\delta}{3D})^M\}$$

$$\cap\{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} < (\frac{\delta}{3D})^M\} \cap \{k \in \mathbb{N} : |a_{q_0} - a|^{q_k} < (\frac{\delta}{3D})^M\}.$$

Then the result follows.

Since the inclusions $m^I_{\mathcal{Z}}(q) \subset l_\infty(q)$ and $m^I_{\mathcal{Z}_0}(q) \subset l_\infty(q)$ are strict so in view of Theorem 2.2.3 we have the following result.

Theorem 3.2.4. The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are nowhere dense subsets of $l_\infty(q)$.

Theorem 3.2.5. The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are not separable.

Proof. We shall prove the result for the space $m^I_{\mathcal{Z}}(q)$. The proof for the other spaces will follow similarly.

Let M be an infinite subset of $\mathbb{N}$ of increasing natural numbers such that $M \in I$. Let

$$q_k = \begin{cases} 1, & \text{if } k \in M, \\ 2, & \text{otherwise.} \end{cases}$$

Let

$$P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{for } \ k \in M \ \ and \ \ x_k = 0, \text{otherwise}\}.$$

Clearly P_0 is uncountable. Consider the class of open balls

$$B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}.$$

Let C_1 be an open cover of $m^I_{\mathcal{Z}}(q)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m^I_{\mathcal{Z}}(q)$. Thus $m^I_{\mathcal{Z}}(q)$ is not separable.

Theorem 3.2.6. Let $G = \sup_k q_k < \infty$ and I an admissible ideal. Then the following are equivalent:

[a] $(x_k) \in \mathcal{Z}^I(q)$;

[b] there exists$(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I;

[c] there exists$(y_k) \in \mathcal{Z}(q)$ and $(x_k) \in \mathcal{Z}^I_0(q)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L|^{q_k}) \geq \epsilon\} \in I$;

[d] there exists a subset

$$K = \{k_1 < k_2....\} \quad \text{of } \mathbb{N},$$

such that $K \in \pounds(I)$ and

$$\lim_{n\to\infty} |x_{k_n} - L|^{q_{k_n}} = 0.$$

Proof.

[a] implies [b].

Let $(x_k) \in \mathcal{Z}^I(q)$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : |x_k^{/} - L|^{q_k} \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : |x_k^{/} - L|^{q_k} \geq t^{-1}\} \in I.$$

Define a sequence (y_k) as

$$y_k = x_k, \quad \text{for all} \quad k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$,

$$y_k = \begin{cases} x_k, & \text{if } |x_k^{/} - L|^{q_k} < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}(q)$ and from the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k^{/} - L|^{q_k}) \geq \epsilon\} \in I,$$

we get $x_k = y_k$, for a.a.k.r.I.

[b] implies [c].

For $(x_k) \in \mathcal{Z}^I(q)$, there exists $(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I. Let

$$K = \{k \in \mathbb{N} : x_k \neq y_k\},$$

then $k \in I$. Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I(q)$ and $y_k \in \mathcal{Z}(q)$.

[c] implies [d].

Suppose [c] holds. Let $\epsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : |z_k^/|^{q_k} \geq \epsilon\} \in I,$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have

$$\lim_{n\to\infty} |x_{k_n}^/ - L|^{q_{k_n}} = 0.$$

[d] implies [a].

Let

$$K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$$

and

$$\lim_{n\to\infty} |x_{k_n}^/ - L|^{q_{k_n}} = 0.$$

Then for any $\epsilon > 0$, and Lemma 3.1.1., we have

$$\{k \in \mathbb{N} : |x_k^/ - L|^{q_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x_k^/ - L|^{q_k} \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I(q)$.

Theorem 3.2.7. Let $h = \inf_k q_k$ and $G = \sup_k q_k$. Then the following results are equivalent.

[a] $G < \infty$ and $h > 0$.

[b] $\mathcal{Z}_0^I(q) = \mathcal{Z}_0^I$.

Proof. Suppose that $G < \infty$ and $h > 0$, then the inequalities

$$min\{1, s^h\} \leq s^{q_k} \leq max\{1, s^G\},$$

hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalence of [a] and [b] is obvious.

Theorem 3.2.8. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m_{\mathcal{Z}_0}^I(q) \supseteq m_{\mathcal{Z}_0}^I(r)$ if and only if $\liminf_{k \in K} \frac{q_k}{r_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. Let $\liminf_{k \in K} \frac{q_k}{r_k} > 0$ and $(x_k) \in m_{\mathcal{Z}_0}^I(r)$. Then there exists $\beta > 0$ such that $q_k > \beta r_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m_{\mathcal{Z}_0}^I(r)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : |x_k|^{r_k} \geq \epsilon\} \in I$$

Let $G_0 = K^c \cup B_0$ then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : |x_k|^{q_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : |x_k|^{\beta r_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in m_{\mathcal{Z}_0}^I(q)$. The converse part of the result follows obviously.

Theorem 3.2.9. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m^I_{\mathcal{Z}_0}(r) \supseteq m^I_{\mathcal{Z}_0}(q)$ if and only if $\lim_{k\in K}\inf \frac{r_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. The proof follows similarly as the proof of Theorem 3.2.8.

Theorem 3.2.10. Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m^I_0(r) = m^I_0(q)$ if and only if $\lim_{k\in K}\inf \frac{q_k}{r_k} > 0$, and $\lim_{k\in K}\inf \frac{r_k}{q_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.

Proof. By combining Theorem 3.2.8 and 3.2.9 we get the required result.

Chapter 4

Zweier I-Convergent Sequence Spaces Defined by Orlicz Function

“Mathematics is a free flow of thoughts and concepts which a mathematicians, in the same way as musician does with the tones of music and a poet with words, puts together into theorems and theories”- Orlicz.

4.1 Introduction

An *Orlicz function* is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

If convexity of M is replaced by $M(x+y) \leq M(x) + M(y)$, then it is called a *Modulus function*, defined and discussed by Nakano [58], Ruckle [62-64].

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Lindenstrauss and Tzafriri [55] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\};$$

which is a Banach space with the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Remark . An Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

For more details on Orlicz sequence spaces we refer to [55], [21-28].

4.2 Main Results

In this chapter we introduce the following classes of sequence spaces:

$$\mathcal{Z}^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^/ - L|}{\rho}) = 0 \text{ for some L and } \rho > 0\},$$

$$\mathcal{Z}_0^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^/|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

$$\mathcal{Z}_\infty^I(M) = \{(x_k) \in \omega : \sup_k M(\frac{|x_k^/|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

Also we denote by

$$m_{\mathcal{Z}}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}^I(M)$$

and

$$m_{\mathcal{Z}_0}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}_0^I(M).$$

Theorem 4.2.1. For any Orlicz function M, the classes of sequences $\mathcal{Z}^I(M), \mathcal{Z}_0^I(M), m_{\mathcal{Z}}^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(M)$. The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I(M)$ and let α, β be scalars. Then there exists positive numbers ρ_1 and ρ_2 such that

$$I - \lim M(\frac{|x_k^/ - L_1|}{\rho_1}) = 0, \text{ for some } L_1 \in \mathbb{C};$$

$$I - \lim M(\frac{|y_k^/ - L_2|}{\rho_2}) = 0, \text{ for some } L_2 \in \mathbb{C}.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : M(\frac{|x_k^{/} - L_1|}{\rho_1}) > \frac{\epsilon}{2}\} \in I, \quad [4.1]$$

$$A_2 = \{k \in \mathbb{N} : M(\frac{|y_k^{/} - L_2|}{\rho_2}) > \frac{\epsilon}{2}\} \in I. \quad [4.2]$$

Let ρ_3 = $\max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex function, we have

$$M(\frac{|(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\leq M(\frac{|\alpha||x_k^{/} - L_1|}{\rho_3}) + M(\frac{|\beta||y_k^{/} - L_2|}{\rho_3})$$

$$\leq M(\frac{|x_k^{/} - L_1|}{\rho_1}) + M(\frac{|y_k^{/} - L_2|}{\rho_2}).$$

Now, by [4.1] and [4.2],

$$\{k \in \mathbb{N} : M(\frac{|(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore

$$(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(M).$$

Hence $\mathcal{Z}^I(M)$ is a linear space.

Theorem 4.2.2. The spaces $m_{\mathcal{Z}}^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are Banach spaces normed by

$$||x_k|| = \inf\{\rho > 0 : \sup_k M(\frac{|x_k|}{\rho}) \leq 1\}.$$

Proof. Proof of this result is easy in view of the existing techniques and therefore is omitted.

Theorem 4.2.3. Let M_1 and M_2 be Orlicz functions that satisfy the $\triangle_2$-condition. Then

[a] $X(M_2) \subseteq X(M_1.M_2)$;

[b] $X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Proof. [a] Let $(x_k) \in \mathcal{Z}_0^I(M_2)$. Then there exists $\rho > 0$ such that

$$I - \lim_k M_2(\frac{|x_k^{/}|}{\rho}) = 0. \tag{4.3}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq t \leq \delta$. Write

$$y_k = M_2(\frac{|x_k^{/}|}{\rho}),$$

and consider

$$\lim_{0 \leq y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k).$$

We have

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) \leq M_1(2). \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k). \tag{4.4}$$

For $(y_k) > \delta$, we have

$$(y_k) < (\frac{y_k}{\delta}) < 1 + (\frac{y_k}{\delta}).$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(y_k) < M_1(1 + (\frac{y_k}{\delta})) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_k}{\delta}).$$

Since M_1 satisfies the $\triangle_2$-condition, we have

$$M_1(y_k) < \frac{1}{2} K(\frac{y_k}{\delta}) M_1(2) + \frac{1}{2} K(\frac{y_k}{\delta}) M_1(2) = K(\frac{y_k}{\delta}) M_1(2).$$

Hence

$$\lim_{y_k>\delta, k\in\mathbb{N}} M_1(y_k) \leq max(1, K\delta^{-1}M_1(2)) \lim_{y_k>\delta, k\in\mathbb{N}} (y_k). \qquad [4.5]$$

From [4.3], [4.4] and [4.5], we have $(x_k) \in \mathcal{Z}_0^I(M_1.M_2)$. Thus

$$\mathcal{Z}_0^I(M_2) \subseteq \mathcal{Z}_0^I(M_1.M_2).$$

The other cases can be proved similarly.

[b] Let

$$(x_k) \in \mathcal{Z}_0^I(M_1) \cap \mathcal{Z}_0^I(M_2).$$

Then there exists $\rho > 0$ such that

$$I - \lim_k M_1(\frac{|x_k^{/}|}{\rho}) = 0$$

and

$$I - \lim_k M_2(\frac{|x_k^{/}|}{\rho}) = 0.$$

The rest of the proof follows from the following equality

$$\lim_{k\in\mathbb{N}}(M_1 + M_2)(\frac{|x_k^{/}|}{\rho}) = \lim_{k\in\mathbb{N}} M_1(\frac{|x_k^{/}|}{\rho}) + \lim_{k\in\mathbb{N}} M_2(\frac{|x_k^{/}|}{\rho}).$$

Therem 4.2.4. The spaces $\mathcal{Z}_0^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(M)$. For $m_{\mathcal{Z}_0}^I(M)$ the result can be proved similarly. Let $(x_k) \in \mathcal{Z}_0^I(M)$. Then there exists $\rho > 0$ such that

$$I - \lim_k M(\frac{|x_k^{/}|}{\rho}) = 0. \qquad [4.6]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [4.6] and the following inequality

$$M(\frac{|\alpha_k x_k^{/}|}{\rho}) \leq |\alpha_k| M(\frac{|x_k^{/}|}{\rho}) \leq M(\frac{|x_k^{/}|}{\rho}) \text{ for all } k \in \mathbb{N}.$$

By Lemma 4.1.1, a sequence space E is solid implies that E is monotone. We have the space $\mathcal{Z}_0^I(M)$ is monotone.

Theorem 4.2.5. The spaces $\mathcal{Z}^I(M)$ and $m_{\mathcal{Z}}^I(M)$ are neither solid nor monotone in general.

Proof. Here we give a counter example.

Let $I = I_\delta$ and $M(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(M)$ of $X(M)$ defined as follows, let $(x_k) \in X(M)$ and let $(y_k) \in X_K(M)$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence x_k defined by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I(M)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(M)$. Thus $\mathcal{Z}^I(M)$ is not monotone. Hence $\mathcal{Z}^I(M)$ is not solid.

Theorem 4.2.6. The spaces $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $M(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \quad and \;\; y_k = k \;\text{ for all } k \in \text{N}.$$

Then $(x_k) \in \mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$, but $(y_k) \notin \mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$. Hence the spaces $\mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$ are not convergence free.

Theorem 4.2.7. The spaces $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(M)$ is a sequence algebra. For the space

$\mathcal{Z}^I(M)$, the result can be proved similarly. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(M)$. Then

$$I - \lim M(\frac{|x_k^/|}{\rho_1}) = 0 \quad \text{for some} \quad \rho_1 > 0$$

and

$$I - \lim M(\frac{|y_k^/|}{\rho_2}) = 0 \quad \text{for some} \quad \rho_2 > 0.$$

Let $\rho = \rho_1.\rho_2 > 0$. Then we can show that

$$I - \lim M(\frac{|(x_k^/.y_k^/)|}{\rho}) = 0.$$

Thus

$$(x_k.y_k) \in \mathcal{Z}_0^I(M).$$

Hence $\mathcal{Z}_0^I(M)$ is a sequence algebra.

Theorem 4.2.8. Let M be an Orlicz function. Then the inclusions $\mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M) \subset \mathcal{Z}_\infty^I(M)$ hold.

Proof. Let $(x_k) \in \mathcal{Z}^I(M)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$I - \lim M(\frac{|x_k^/ - L|}{\rho}) = 0.$$

We have

$$M(\frac{|x_k^/|}{2\rho}) \leq \frac{1}{2}M(\frac{|x_k^/ - L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho}).$$

Taking supremum over k both sides we get

$$(x_k) \in \mathcal{Z}_\infty^I(M).$$

The inclusion

$$\mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M)$$

is obvious.

Theorem 4.2.9. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $M(x) = x$ for all $x \in [0, \infty)$. If

$$x_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in \mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M)$, by lemma 3.1.8. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \text{otherwise.} \end{cases}$$

is a permutation on $\mathbb{N}$, but $(x_{\pi(k)}) \notin \mathcal{Z}^I(M)$ and $(x_{\pi(k)}) \notin \mathcal{Z}_0^I(M)$. Hence $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$are not symmetric.

Chapter 5

On Some Zweier I-Convergent Sequence Spaces Defined by a Modulus Function

"Good mathematicians see analogies between theorems or theories, the very best ones see analogies between analogies"- Banach.

5.1 Introduction

Ruckle[62-64] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle[62] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[62-64] proved that, for any modulus f,

$$X(f) \subset \ell_1 \text{ and } X(f)^\alpha = \ell_\infty$$

where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[62]).}$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in[16]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling[14-15], Köthe[50], Kolk[51-52] and Ruckle[29-31].

In this chapter we introduce the following class of sequence spaces.

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \text{there is } L \in \mathbb{C} \text{ such that}$$
$$\text{for } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \text{for a given } \varepsilon > 0, \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M\} \in I, \text{ for each fixed M>0}\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty(f) \cap \mathcal{Z}_0^I(f).$$

5.2 Main Results

Theorem 5.2.1. For any modulus function f, the classes of sequences $\mathcal{Z}^I(f), \mathcal{Z}_0^I(f), m_{\mathcal{Z}}^I(f)$ and $m_{\mathcal{Z}_0}^I(f)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(f)$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I(f)$ and let α, β be scalars. Then

$$I - \lim f(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C} \quad ;$$
$$I - \lim f(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C} \quad ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : f(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{5.1}$$

$$A_2 = \{k \in \mathbb{N} : f(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{5.2}$$

Since f is a modulus function, we have

$$f(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) \leq f(|\alpha||x_k - L_1|) + f(|\beta||y_k - L_2|)$$
$$\leq f(|x_k - L_1|) + f(|y_k - L_2|)$$

Now, by [5.1] and [5.2], $\{k \in$ N: $f(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(f)$. Hence $\mathcal{Z}^I(f)$ is a linear space.

We state the following result without proof in view of Theorem 5.2.1.

Theorem 5.2.2. The spaces $m^I_{\mathcal{Z}}(f)$ and $m^I_{\mathcal{Z}_0}(f)$ are normed linear spaces, normed by

$$||x_k||_* = \sup_k f(|x_k|). \tag{5.3}$$

Theorem 5.2.3. A sequence $x = (x_k) \in m^I_{\mathcal{Z}}(f)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(f). \tag{5.4}$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I_{\mathcal{Z}}(f). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(f)$.

Conversely, suppose that $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(f)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(f)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}}(f) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(f)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(f)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(f)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(f)$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(f)$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

Theorem 5.2.4. Let f and g be modulus functions that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I$, $\mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ and $m^I_{\mathcal{Z}_0}$ etc, then the following assertions hold.
(a) $X(g) \subseteq X(f.g)$,
(b) $X(f) \cap X(g) \subseteq X(f + g)$.

Proof. (a) Let $(x_k) \in \mathcal{Z}^I_0(g)$. Then

$$I - \lim_k g(|x_k|) = 0. \qquad [5.5]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$. Write $y_k = g(|x_k|)$ and consider $\lim_k f(y_k) = \lim_k f(y_k)_{y_k < \delta} + \lim_k f(y_k)_{y_k > \delta}$. We have

$$\lim_k f(y_k) \leq f(2) \lim_k (y_k) \qquad [5.6]$$

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since f is non-decreasing, it follows that

$$f(y_k) < f(1 + \frac{y_k}{\delta}) < \frac{1}{2}f(2) + \frac{1}{2}f(\frac{2y_k}{\delta})$$

Since f satisfies the $\triangle_2$-condition, we have

$$f(y_k) < \frac{1}{2}K\frac{y_k}{\delta}f(2) + \frac{1}{2}K\frac{y_k}{\delta}f(2) = K\frac{y_k}{\delta}f(2)$$

Hence

$$\lim_k f(y_k) \leq max(1, K)\delta^{-1}f(2)\lim_k(y_k). \quad [5.7]$$

From [5.5], [5.6] and [5.7] we have $(x_k) \in \mathcal{Z}_0^I(f.g)$.

Thus $\mathcal{Z}_0^I(g) \subseteq \mathcal{Z}_0^I(f.g)$. The other cases can be proved similarly.

(b) Let $(x_k) \in \mathcal{Z}_0^I(f) \cap \mathcal{Z}_0^I(g)$. Then

$$I - \lim_k f(|x_k|) = 0 \; and \; I - \lim_k g(|x_k|) = 0$$

The rest of the proof follows from the following equality

$$\lim_k(f + g)(|x_k|) = \lim_k f(|x_k|) + \lim_k g(|x_k|).$$

Corollary 5.2.5. $X \subseteq X(f)$ for X =$\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 5.2.6. The spaces $\mathcal{Z}_0^I(f)$ and $m_{\mathcal{Z}_0}^I(f)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(f)$. Let $(x_k) \in \mathcal{Z}_0^I(f)$. Then

$$I - \lim_k f(|x_k|) = 0. \quad [5.8]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [5.8] and the following inequality

$$f(|\alpha_k x_k|) \leq |\alpha_k| f(|x_k|) \leq f(|x_k|) \; for \; all \; k \in \mathbb{N}.$$

That the space $\mathcal{Z}_0^I(f)$ is monotone follows from the Lemma 5.1.1. For $m_{\mathcal{Z}_0}^I(f)$ the result can be proved similarly.

Theorem 5.2.7. The spaces $\mathcal{Z}^I(f)$ and $m_{\mathcal{Z}}^I(f)$ are neither solid nor monotone in general .

Proof. Here we give a counter example. Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows.

Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), & \text{if k is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I(f)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(f)$. Thus $\mathcal{Z}^I(f)$ is not monotone. Hence $\mathcal{Z}^I(f)$ is not solid.

Theorem 5.2.8. The spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(f)$ is a sequence algebra. Let
$(x_k), (y_k) \in \mathcal{Z}_0^I(f)$. Then

$$I - \lim f(|x_k|) = 0$$

and

$$I - \lim f(|y_k|) = 0$$

Then we have

$$I - \lim f(|(x_k.y_k)|) = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I(f)$ is a sequence algebra. For the space $\mathcal{Z}_0^I(f)$, the result can be proved similarly.

Theorem 5.2.9. The spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \text{ and } y_k = k \text{ for all } k \in \mathbb{N}$$

Then $(x_k) \in \mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$, but $(y_k) \notin \mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$. Hence the spaces $\mathcal{Z}_0^I(f)$ and $\mathcal{Z}_0^I(f)$ are not convergence free.

Theorem 5.2.10. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f(x) = x$ for all $x \in [0, \infty)$. If

$$x_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.22 $(x_k) \in \mathcal{Z}_0^I(f) \subset \mathcal{Z}^I(f)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin \mathcal{Z}^I(f)$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I(f)$. Hence $\mathcal{Z}^I(f)$ and $\mathcal{Z}_0^I(f)$ are not symmetric.

Theorem 5.2.11. Let f be a modulus function. Then $\mathcal{Z}_0^I(f) \subset \mathcal{Z}^I(f) \subset \mathcal{Z}_\infty^I(f)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f(|x_k - L|) = 0$$

We have $f(|x_k|) \leq \frac{1}{2}f(|x_k - L|) + f\frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}^I_\infty(f)$. The inclusion $\mathcal{Z}^I_0(f) \subset \mathcal{Z}^I(f)$ is obvious.

Theorem 5.2.12. The function $\hbar : m^I_{\mathcal{Z}}(f) \to \mathbb{R}$ is the Lipschitz function, where $m^I_{\mathcal{Z}}(f) = \mathcal{Z}^I_\infty(f) \cap \mathcal{Z}^I(f)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}(f)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(f),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(f).$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}(f)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 5.2.13. If $x, y \in m^I_{\mathcal{Z}}(f)$, then $(x.y) \in m^I_{\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I_{\mathcal{Z}}(f),$$

$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I_{\mathcal{Z}}(f).$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)|$$
$$\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \qquad [5.9]$$

As $m^I_{\mathcal{Z}}(f) \subseteq \mathcal{Z}^I_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn [5.9] we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m^I(f)$. Hence $(x.y) \in m^I_{\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{\mathcal{Z}_0}(f)$ the result can be proved similarly.

Chapter 6

Zweier I-Convergent Sequence Spaces Defined by a Sequence of Modulii

"The essence of mathematics lies in its freedom"- Cantor.

6.1 Introduction

Recently Khan and Ebadullah[31] introduced the following classes of sequences

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C} \ \} \in I\},$$

$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f).$$

In this chapter we introduce the following class of sequence spaces.

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C} \ \} \in I\},$$

$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty(F) \cap \mathcal{Z}^I(F)$$

and

$$m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty(F) \cap \mathcal{Z}_0^I(F).$$

6.2 Main Results

Theorem 6.2.1. For a sequence of modulii $F = (f_k)$, the classes of sequences $\mathcal{Z}^I(F), \mathcal{Z}_0^I(F), m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C} \quad ;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C} \quad ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \qquad [6.1]$$

$$A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \qquad [6.2]$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|)$$
$$\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|)$$

Now, by [6.1] and [6.2], $\{k \in \mathbb{N}$: $f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\}$ $\subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$. Hence $\mathcal{Z}^I(F)$ is a linear space.

We state the following result without proof in view of Theorem 6.2.1.

Theorem 6.2.2. The spaces $m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are normed linear spaces, normed by

$$||x_k||_* = \sup_k f_k(|x_k|). \qquad [6.3]$$

Theorem 6.2.3. A sequence $x = (x_k) \in m^I_{\mathcal{Z}}(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F). \qquad [6.4]$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I_{\mathcal{Z}}(F). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$.Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}}(F) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(F)$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \ldots.. \supseteq I_k \supseteq \ldots\ldots\ldots.$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(F)$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$.

Theorem 6.2.4. Let (f_k) and (g_k) be modulus functions for some fixed k that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I, \mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ and $m^I_{\mathcal{Z}_0}$ etc, then the following assertions hold.
(a) $X(g_k) \subseteq X(f_k.g_k)$,
(b) $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

Proof. (a) Let $(x_n) \in \mathcal{Z}^I_0(g_k)$. Then

$$I - \lim_n g_k(|x_n|) = 0 \qquad [6.5]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 < t < \delta$. Write $y_n = g_k(|x_n|)$ and consider $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n<\delta} + \lim_n f_k(y_n)_{y_n>\delta}$. We have

$$\lim_n f_k(y_n) \leq f_k(2) \lim_n (y_n). \qquad [6.6]$$

For $y_n > \delta$, we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since f_k is non-decreasing, it follows that

$$f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2y_n}{\delta})$$

Since f_k satisfies the $\triangle_2$-condition, we have

$$f_k(y_n) < \frac{1}{2} K \frac{y_n}{\delta} f_k(2) + \frac{1}{2} K \frac{y_n}{\delta} f_k(2) = K \frac{y_n}{\delta} f_k(2)$$

Hence

$$\lim_n f_k(y_n) \leq \text{max(1,K)} \delta^{-1} f_k(2) \lim_n (y_n). \qquad [6.7]$$

From [6.5], [6.6] and [6.7], we have $(x_n) \in \mathcal{Z}_0^I(f_k.g_k)$.

Thus $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k.g_k)$. The other cases can be proved similarly.

(b) Let $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$. Then

$$I - \lim_n f_k(|x_n|) = 0 \; and \; I - \lim_n g_k(|x_n|) = 0$$

The rest of the proof follows from the following equality

$$\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|).$$

Corollary 6.2.5. $X \subseteq X(f_k)$ for some fixed k and X =$\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 6.2.6. The spaces $\mathcal{Z}_0^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(F)$. Let $(x_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim_k f_k(|x_k|) = 0 \qquad [6.8]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [6.8] and the following inequality

$$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|) \; for \; all \; k \in \mathbb{N}.$$

That the space $\mathcal{Z}_0^I(F)$ is monotone follows from the Lemma 6.1.1. For $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly.

Theorem 6.2.7. The spaces $\mathcal{Z}^I(F)$ and $m_{\mathcal{Z}}^I(F)$ are neither solid nor monotone in general .

Proof. Here we give a counter example. Let $I = I_\delta$ and $f_k(x) = x^2$ for some fixed k and for all $x \in [0, \infty)$. Consider the K-step space $X_K(f_k)$ of X defined as follows.

Let $(x_n) \in X$ and let $(y_n) \in X_K$ be such that

$$(y_n) = \begin{cases} (x_n), & \text{if n is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence (x_n) defined by $(x_n) = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(F)$. Thus $\mathcal{Z}^I(F)$ is not monotone. Hence $\mathcal{Z}^I(F)$ is not solid.

Theorem 6.2.8. The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(F)$ is a sequence algebra. Let
$(x_k), (y_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0$$

Then we have

$$I - \lim f_k(|(x_k.y_k)|) = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I(F)$ is a sequence algebra. For the space $\mathcal{Z}^I(F)$, the result can be proved similarly.

Theorem 6.2.9. The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $f_k(x) = x^3$ for some fixed k and for all $x \in [0, \infty)$. Consider the sequence (x_n) and (y_n) defined by

$$x_n = \frac{1}{n} \quad and \;\; y_n = n \;\; \text{for all } n \in \mathbb{N}$$

Then $(x_n) \in \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$, but $(y_n) \notin \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$. Hence the spaces $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free.

Theorem 6.2.10. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f_k(x) = x$ for some fixed k and for all $x \in [0, \infty)$.

If

$$x_n = \begin{cases} 1, & \text{for } n \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.22 $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(n) = \begin{cases} \phi(n), & \text{for } n \in K, \\ \psi(n), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(n)} \notin \mathcal{Z}^I(F)$ and $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$. Hence $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.

Theorem 6.2.11. $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f_k(|x_k - L|) = 0$$

We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I(F)$. The inclusion $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ is obvious.

Theorem 6.2.12. The function $\hbar : m_{\mathcal{Z}}^I(F) \to \mathbb{R}$ is the Lipschitz function, where $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x - y||_*\} \in m^I_{\mathcal{Z}}(F).$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}(F)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly.

Theorem 6.2.13. If $x, y \in m^I_{\mathcal{Z}}(F)$, then $(x.y) \in m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I_{\mathcal{Z}}(F),$$

$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I_{\mathcal{Z}}(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)| \qquad [6.9] \end{aligned}$$

As $m^I_{\mathcal{Z}}(F) \subseteq \mathcal{Z}^I_\infty(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn[6.9] we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly.

Chapter 7

On Certain Class of Zweier I-Convergent Sequence Spaces

7.1 Introduction

Theorem 7.1.1. [68, Theorem 2.1] The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$||x||_{\mathcal{Z}} = ||x||_{\mathcal{Z}_0} = ||Z^p x||_c.$$

Theorem 7.1.2. [68, Theorem 2.2] The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.

Theorem 1.3. [68, Theorem 2.3] The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.

7.2 Main Results

Recently Šalát, Tripathy and Ziman[65-66] introduced the following sequence spaces

$$c_0^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k| \geq \epsilon\} \in I\},$$
$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C} \ \},$$
$$\ell_\infty^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k| \geq M\} \in I, \text{ for each fixed M>0}\}.$$

Analogous to Kostyrko, Šalát and Wilczyński[12], Šalát Tripathy and Ziman[65-66], Khan and Ebadullah[29,31,37,38] introduced the following classes of sequences.

$$\mathcal{Z}_0^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = 0\} \in I\},$$

"If I feel unhappy, I do mathematics to become happy. If I feel happy, I do mathematics to keep happy." -Paul Turan

$$\mathcal{Z}^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = L, \text{ for some L}\} \in I\},$$

$$\mathcal{Z}^I_\infty = \{x = (x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

In [27] for $q = (q_k)$ a sequence of positive reals

$$\mathcal{Z}^I_0(q) = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x|^{q_k} \geq \epsilon\} \in I\},$$

$$\mathcal{Z}^I(q) = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : |Z^p x - L|^{q_k} \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\ \},$$

$$\mathcal{Z}^I_\infty(q) = \{x = (x_k) \in \omega : \sup_k |Z^p x|^{q_k} < \infty\}.$$

In [8] for an Orlicz function M and $Z^p x = x^/$

$$\mathcal{Z}^I_0(M) = \{x = (x_k) \in \omega : I - \lim M(\frac{|x^/_k|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

$$\mathcal{Z}^I(M) = \{x = (x_k) \in \omega : I - \lim M(\frac{|x^/_k - L|}{\rho}) = 0 \text{ for some L and } \rho > 0\},$$

$$\mathcal{Z}^I_\infty(M) = \{x = (x_k) \in \omega : \sup_k M(\frac{|x^/_k|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

In [29] for a modulus function f

$$\mathcal{Z}^I_0(f) = \{(x_k) \in \omega : \text{for a given } \varepsilon > 0, \{k \in \mathbb{N} : f(|x^/_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \text{there is } L \in \mathbb{C} \text{ such that}$$
$$\text{for } \varepsilon > 0, \{k \in \mathbb{N} : f(|x^/_k - L|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}^I_\infty(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x^/_k|) \geq M\} \in I, \text{ for each fixed M>0}\}.$$

In [34] for a sequence of modulii $F = (f_k)$

$$\mathcal{Z}^I_0(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x^/_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x^/_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$

$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k^/|) \geq M, \text{for each fixed M>0}\} \in I\}.$$

Here we give the canonical inclusion relations

Result 7.2.1. $c_0^I \subset c^I \subset \ell_\infty^I$.(See[41,57,58]).

Result 7.2.2. $\mathcal{Z}_0^I \subset \mathcal{Z}^I \subset \mathcal{Z}_\infty^I$.(See[12]).

Result 7.2.3. $\mathcal{Z}_0^I(q) \subset \mathcal{Z}^I(q) \subset \mathcal{Z}_\infty^I(q)$.(See[27]).

Result 7.2.4. $\mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M) \subset \mathcal{Z}_\infty^I(M)$.(See[35]).

Result 7.2.5. $\mathcal{Z}_0^I(f) \subset \mathcal{Z}^I(f) \subset \mathcal{Z}_\infty^I(f)$.(See[29]).

Result 7.2.6. $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

Chapter 8

Zweier I-Convergent Double Sequence Spaces

8.1 Introduction

At the initial stage the notion of I-convergence was introduced by Kostyrko, Šalát and Wilczyński[48]. Later on it was studied by Šalát, Tripathy and Ziman[65], Demirci [10] and many others. I-convergence is a generalization of Statistical Convergence.

Now we have a list of some basic definitions used in the chapter:

Definition 8.1. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}), where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \epsilon, \qquad for\ all\ i, j \geq N \ \ (see\ [6,7,8])$$

Definition 8.2. A double sequence $(x_{ij}) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_{ij} = L$.

Definition 8.3. A double sequence $(x_{ij}) \in \omega$ is said to be I-null if L = 0. In this case we write

$$I - \lim x_{ij} = 0.$$

Definition 8.4. A double sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for

"Example is the school of mankind, and they will learn at no other."-Edmund Burke every $\epsilon > 0$ there exist numbers m = m(ϵ), n= n(ϵ) such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I.$$

Definition 8.5. A double sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}.$$

Definition 8.6. A double sequence space E is said to be solid or normal if $(x_{ij}) \in$ E implies $(\alpha_{ij}x_{ij}) \in$ E for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all (i, j) $\in \mathbb{N} \times \mathbb{N}$.

Definition 8.7. A double sequence space E is said to be monotone if it contains the canonical preimages of its stepspaces.

Definition 8.8. A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 8.9. A double sequence space E is said to be a sequence algebra if $(x_{ij}.y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 8.10. A double sequence space E is said to be symmetric if $(x_{ij}) \in E$ implies $(x_{\pi(ij)}) \in E$, where π is a permutation on $\mathbb{N} \times \mathbb{N}$.

In this Chapter we introduce the following classes of sequence space:

$$ {}_2\mathcal{Z}^I = \{x = (x_{ij}) \in {}_2\omega : I - \lim Z^p x = L \text{ for some L} \in \mathbb{C}\ \}$$

$$ {}_2\mathcal{Z}_0^I = \{x = (x_{ij}) \in {}_2\omega : I - \lim Z^p x = 0\}$$

$$ {}_2\mathcal{Z}_\infty^I = \{x = (x_{ij}) \in {}_2\omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : \text{there exist } M > 0, |Z^p x| \geq M\} \in I\}$$

$$ {}_2\mathcal{Z}_\infty = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} |Z^p x| < \infty\}$$

We also denote the multiplier double sequence spaces as

$$ {}_2m^I_{\mathcal{Z}} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I \quad and \quad {}_2m^I_{\mathcal{Z}_0} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I_0. $$

8.2 Main Results

Theorem 8.2.1. The classes of sequences ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0$, ${}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I$ and let α, β be scalars. Then

$$ I - \lim |x_{ij} - L_1| = 0, \text{ for some } L_1 \in \mathbb{C}; $$

$$ I - \lim |y_{ij} - L_2| = 0, \text{ for some } L_2 \in \mathbb{C}. $$

That is for a given $\epsilon > 0$, we have

$$ A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L_1| > \frac{\epsilon}{2}\} \in I, \tag{8.1} $$

$$ A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L_2| > \frac{\epsilon}{2}\} \in I. \tag{8.2} $$

We have

$$ \begin{aligned} |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| &\le |\alpha|(|x_{ij} - L_1|) + |\beta|(|y_{ij} - L_2|) \\ &\le |x_{ij} - L_1| + |y_{ij} - L_2|. \end{aligned} $$

Now, by [8.1] and [8.2],

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2. $$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I$. Hence ${}_2\mathcal{Z}^I$ is a linear space.

We state the following result without proof in view of Theorem 2.1.

Theorem 8.2.2. The spaces ${}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$ are normed linear spaces, normed by

$$||x_{ij}||_* = \sup_{i,j} |x_{ij}|. \qquad [8.3]$$

Theorem 8.2.3. A sequence $x = (x_{ij}) \in {}_2m^I_{\mathcal{Z}}$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon = (m, n) \in \mathbb{N} \times \mathbb{N}$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}} \qquad [8.4]$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| < \frac{\epsilon}{2}\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon = (m, n) \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_{ij}| \leq |x_{N_\epsilon} - L| + |L - x_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i, j) \in B_\epsilon$. Hence
$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$.

Conversely, suppose that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$. That is

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_k - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$$

for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i, j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in {}_2m^I_{\mathcal{Z}}$ as well as $C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in J\} \in {}_2m^I_{\mathcal{Z}}$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in I_k\} \in {}_2m^I_{\mathcal{Z}}$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $\xi = I - \lim x$, that is $L = I - \lim x$.

Theorem 8.2.4. Let I be an admissible ideal. Then the following are equivalent.

(a) $(x_{ij}) \in {}_2\mathcal{Z}^I$;

(b) there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I;

(c) there exists$(y_{ij}) \in {}_2\mathcal{Z}$ and $(z_{ij}) \in {}_2\mathcal{Z}^I_0$ such that $x_{ij} = y_{ij} + z_{ij}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ and

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L| \geq \epsilon\} \in I;$$

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

Proof. (a) implies (b). Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists $L \in \mathbb{C}$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

Let (m_t, n_t) be an increasing sequence with $(m_t, n_t) \in \mathbb{N} \times \mathbb{N}$ such that

$$\{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_{ij}) as

$$y_{ij} = x_{ij}, \text{ for all } (i,j) \leq (m_1, n_1).$$

For $(m_t, n_t) < (i,j) \leq (m_{t+1}, n_{t+1}) \quad for \quad t \in \mathbb{N}$.

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } |x_{ij} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_{ij}) \in {}_2\mathcal{Z}$ and form the following inclusion

$$\{(i,j) \leq (m_t, n_t) : x_{ij} \neq y_{ij}\} \subseteq \{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \epsilon\} \in I.$$

We get $x_{ij} = y_{ij}$, for a.a.k.r.I.

(b) implies (c). For $(x_{ij}) \in {}_2\mathcal{Z}^I$, there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I. Let $K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\}$, then $K \in I$. Define a sequence (z_{ij}) as

$$z_{ij} = \begin{cases} x_{ij} - y_{ij}, & \text{if } (i,j) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_{ij} \in {}_2\mathcal{Z}_0^I$ and $y_{ij} \in {}_2\mathcal{Z}$.

(c) implies (d). Let $P_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |z_{ij}| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I).$$

Then we have $\lim_{n \to \infty} |x_{(i_n, j_n)} - L| = 0$.

(d) implies (a). Let $K = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I)$ and $\lim_{n \to \infty} |x_{(i_n, j_n)} - L| = 0$. Then for any $\epsilon > 0$, and Lemma 1.17, we have

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \subseteq K^c \cup \{(i,j) \in K : |x_{ij} - L| \geq \epsilon\}.$$

Thus $(x_{ij}) \in {}_2\mathcal{Z}^I$.

Theorem 8.2.5. The inclusions ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I \subset {}_2\mathcal{Z}_\infty^I$ hold and are proper.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists L $\in \mathbb{C}$ such that

$$I - \lim |x_{ij} - L| = 0$$

We have $|x_{ij}| \leq \frac{1}{2}|x_{ij} - L| + \frac{1}{2}|L|$. Taking the supremum over (i, j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I$. The inclusion ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$ is obvious. The strict inclusion is also trivial.

Theorem 8.2.6. The function $\hbar : {}_2m_\mathcal{Z}^I \to \mathbb{R}$ is the Lipschitz function, where ${}_2m_\mathcal{Z}^I = {}_2\mathcal{Z}^I \cap {}_2\mathcal{Z}_\infty$, and hence uniformly continuous.

Proof. Let $x, y \in {}_2m_\mathcal{Z}^I$, $x \neq y$. Then the sets

$$A_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < ||x - y||_*\} \in {}_2m_\mathcal{Z}^I,$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < ||x - y||_*\} \in {}_2m_\mathcal{Z}^I.$$

Hence also $B = B_x \cap B_y \in {}_2m_\mathcal{Z}^I$, so that $B \neq \phi$. Now taking (i,j) in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.7. If $x, y \in {}_2m_\mathcal{Z}^I$, then $(x.y) \in {}_2m_\mathcal{Z}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x - \hbar(x)| < \epsilon\} \in {}_2m_{\mathcal{Z}}^I,$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y - \hbar(y)| < \epsilon\} \in {}_2m_{\mathcal{Z}}^I.$$

Now,

$$|x.y - \hbar(x)\hbar(y)| = |x.y - x\hbar(y) + x\hbar(y) - \hbar(x)\hbar(y)|$$

$$\leq |x||y - \hbar(y)| + |\hbar(y)||x - \hbar(x)| \qquad [8.5]$$

As ${}_2m_{\mathcal{Z}}^I \subseteq {}_2\mathcal{Z}_\infty$, there exists an $M \in \mathbb{R}$ such that $\hbar|x| < M$ and $|\hbar(y)| < M$. Using eqn[8.5] we get

$$|x.y - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i,j) \in B_x \cap B_y \in {}_2m_{\mathcal{Z}}^I$. Hence $(x.y) \in {}_2m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.8. The spaces ${}_2\mathcal{Z}_0^I$ and ${}_2m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I$. Let $(x_{ij}) \in \mathcal{Z}_0^I$. Then

$$I - \lim_k |x_{ij}| = 0 \qquad [8.6]$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [8.6] and the following inequality

$$|\alpha_{ij}x_{ij}| \leq |\alpha_{ij}||x_{ij}| \leq |x_{ij}| \; for\ all\ (i,j) \in \mathbb{N} \times \mathbb{N}.$$

That the space ${}_2\mathcal{Z}_0^I$ is monotone follows from the Lemma 1.16. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.9. If I is not maximal, then the space ${}_2\mathcal{Z}^I$ is neither solid nor monotone.

Proof. Here we give a counter example. Let $(x_{ij}) = 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I$. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} \times \mathbb{N} - K \notin I$. Define the sequence

$$(y_{ij}) = \begin{cases} (x_{ij}), & \text{if } (i, j) \in K, \\ 0, & otherwise. \end{cases}$$

Then (y_{ij}) belongs to the canonical preimage of K-step space of ${}_2\mathcal{Z}^I$ but $(y_{ij}) \notin {}_2\mathcal{Z}^I$. Hence ${}_2\mathcal{Z}^I$ is not monotone.

Theorem 8.2.10. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are sequence algebras.

Proof. We prove that ${}_2\mathcal{Z}_0^I$ is a sequence algebra. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I$. Then

$$I - \lim |x_{ij}| = 0 \quad and \quad I - \lim |y_{ij}| = 0$$

Then we have $I - \lim |(x_{ij}.y_{ij})| = 0$. Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I$. Hence ${}_2\mathcal{Z}_0^I$ is a sequence algebra. For the space ${}_2\mathcal{Z}^I$, the result can be proved similarly.

Theorem 8.2.11. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i.j} \text{ and } y_{ij} = i.j \text{ for all (i,j)} \in \mathbb{N} \times \mathbb{N}$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$. Hence the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free.

Theorem 8.2.12. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Proof. Let $A \in I$ be infinite. If

$$x_{ij} = \begin{cases} 1, & \text{for } i\ ,\ j \in A, \\ 0, & otherwise. \end{cases}$$

Then $x_{ij} \in {}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{(\pi(m)\pi(n))} \notin {}_2\mathcal{Z}^I$ and $x_{(\pi(m)\pi(n))} \notin {}_2\mathcal{Z}_0^I$. Hence ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Theorem 8.2.13. The sequence spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are linearly isomorphic to the spaces ${}_2c^I$ and ${}_2c_0^I$ respectively, i.e ${}_2\mathcal{Z}^I \cong {}_2c^I$ and ${}_2\mathcal{Z}_0^I \cong {}_2c_0^I$.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I$ and ${}_2c^I$. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces ${}_2\mathcal{Z}^I$ and ${}_2c^I$. Define a map $T : {}_2\mathcal{Z}^I \longrightarrow {}_2c^I$ such that $x \to x^{'} = Tx$

$$T(x_{ij}) = px_{ij} + (1-p)x_{(i-1)(j-1)} = x_{ij}^{'}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x_{ij}^{'} \in {}_2c^I$ and define the sequence $x = x_{ij}$ by

$$x_{ij} = M \sum_{r=0}^{i} \sum_{s=0}^{j} (-1)^{(i-r)(j-s)} N^{(i-r)(j-s)} x_{ij}^{'}$$

for $(i,j) \in \mathbb{N} \times \mathbb{N}$ and where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{(i,j)\to\infty} px_{ij} + (1-p)x_{(i-1)(j-1)} =$$

$$p \lim_{(i,j)\to\infty} M \sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{(i-r)(j-s)}N^{(i-r)(j-s)}x'_{ij}$$

$$+(1-p)\lim_{(i,j)\to\infty} M \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}(-1)^{(i-1-r)(j-1-s)}N^{(i-1-r)(j-1-s)}x'_{(i-1)(j-1)}$$

$$= \lim_{(i,j)\to\infty} x'_{ij}$$

which shows that $x \in {}_2\mathcal{Z}^I$. Hence T is a linear bijection. Also we have $||x||_* = ||Z^p x||_c$. Therefore

$$||x||_* = \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |px_{ij} + (1-p)x_{(i-1)(j-1)}|$$

$$= \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |pM \sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{(i-r)(j-s)}N^{(i-r)(j-s)}x'_{ij}$$

$$+(1-p)M\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}(-1)^{(i-1-r)(j-1-s)}N^{(i-1-r)(j-1-s)}x'_{(i-1)(j-1)}|$$

$$= \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |x'_{ij}| = ||x'||_{\,{}_2c^I}.$$

Hence ${}_2\mathcal{Z}^I \cong {}_2c^I$.

Chapter 9

Zweier I-Convergent Double Sequence Spaces Defined by a Modulus Function

9.1 Introduction

An *Orlicz function* is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.(see[4,47]). If the convexity of the regular function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called as *Modulus function*. This function was introduced by Nakano[58]. Ruckle[64] and Maddox[56] further investigated the modulus function with applications to sequence spaces.

In this chapter we introduce the following class of sequence spaces:

$$_2\mathcal{Z}^I(f) = \{(x_{ij}) \in {}_2\omega : I - \lim f(|x^{'}_{ij} - L|) = 0, \text{ for some } L \in \mathbb{C} \ \},$$

$$_2\mathcal{Z}^I_0(f) = \{(x_{ij}) \in {}_2\omega : I - \lim f(|x^{'}_{ij}|) = 0\},$$

$$_2\mathcal{Z}^I_\infty(f) = \{(x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} :$$

$$\text{there exist } K > 0 : f(|x^{'}_{ij}|) \geq K \in I\}.$$

$$_2\mathcal{Z}_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} f(|x^{'}_{ij}|) < \infty\}$$

Throughout we denote

$$m^I_{_2\mathcal{Z}}(f) = {}_2\mathcal{Z}^I_\infty(f) \cap {}_2\mathcal{Z}(f) \quad and \quad m^I_{_2\mathcal{Z}_0}(f) = {}_2\mathcal{Z}^I_\infty(f) \cap {}_2\mathcal{Z}_0(f).$$

Throughout the article, for the sake of convenience we will denote by $Z^p(x_{ij}) = x^{'}, Z^p(y_{ij}) = y^{'}, Z^p(z_{ij}) = z^{'}$ for $x, y, z \in \omega$.

"Under the leadership of our dear masters Banach and Steinhauss we were practicing in Lwów intricacies of mathematics"- Orlicz-1968.

9.2 Main Results

Theorem 9.2.1. For any modulus function f, the classes of sequences ${}_2\mathcal{Z}^I(f),\ {}_2\mathcal{Z}_0^I(f), m^I_{{}_2\mathcal{Z}}(f)$ and $m^I_{{}_2\mathcal{Z}_0}(f)$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(f)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I(f)$ and let α, β be scalars. Then

$$I - \lim f(|x'_{ij} - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C} \ ;$$

$$I - \lim f(|y'_{ij} - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C} \ ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x'_{ij} - L_1|) > \frac{\epsilon}{2}\} \in I, \qquad [9.1]$$

$$A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|y'_{ij} - L_2|) > \frac{\epsilon}{2}\} \in I. \qquad [9.2]$$

Since f is a modulus function, we have

$$\begin{aligned} f(|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)) &\leq f(|\alpha||x'_{ij} - L_1|) + f(|\beta||y'_{ij} - L_2|) \\ &\leq f(|x'_{ij} - L_1|) \ + \ f(|y'_{ij} - L_2|) \end{aligned}$$

Now, by [9.1] and [9.2],

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I(f)$. Hence ${}_2\mathcal{Z}^I(f)$ is a linear space.

We state the following result without proof in view of Theorem 2.1.

Theorem 9.2.2. The spaces $m^I_{2\mathcal{Z}}(f)$ and $m^I_{2\mathcal{Z}_0}(f)$ are normed linear spaces, normed by

$$||x^{'}_{ij}||_{*} = \sup_{i,j} f(|x^{'}_{ij}|). \qquad [9.3]$$

Theorem 9.2.3. A sequence $x = (x_{ij}) \in m^I_{2\mathcal{Z}}(f)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{k} - x^{'}_{N_\epsilon}|) < \epsilon\} \in \ m^I_{2\mathcal{Z}}(f) \qquad [9.4]$$

Proof. Suppose that $L = I - \lim x^{'}$. Then

$$B_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{'}_{ij} - L| < \frac{\epsilon}{2}\} \in \ m^I_{2\mathcal{Z}}(f). \text{ For } all\ \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x^{'}_{N_\epsilon} - x^{'}_{ij}| \leq |x^{'}_{N_\epsilon} - L| + |L - x^{'}_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i,j) \in B_\epsilon$. Hence

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{ij} - x^{'}_{N_\epsilon}|) < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

Conversely, suppose that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{ij} - x^{'}_{N_\epsilon}|) < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

That is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{'}_{ij} - x^{'}_{N_\epsilon}| < \epsilon\} \in m^I_{2\mathcal{Z}}(f)$$

for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x^{'}_{ij} \in [x^{'}_{N_\epsilon} - \epsilon, x^{'}_{N_\epsilon} + \epsilon]\} \in m^I_{2\mathcal{Z}}(f) \text{ for } all\ \epsilon > 0.$$

Let $J_\epsilon = [x^{'}_{N_\epsilon} - \epsilon, x^{'}_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{2\mathcal{Z}}(f)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{2\mathcal{Z}}(f)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{2\mathcal{Z}}(f)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x'_{ij} \in J\} \in m^I_{2\mathcal{Z}}(f)$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_{ij} \supseteq ....$$

with the property that $diam\ I_{ij} \leq \frac{1}{2}\ diam\ I_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x'_{ij} \in I_{ij}\} \in m^I_{2\mathcal{Z}}(f)$ for (k=1,2,3,4,......).

Then there exists a $\xi \in \cap I_k$ where $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $\xi = I - \lim x'$. So that $f(\xi) = I - \lim f(x')$, that is $L = I - \lim f(x')$.

Theorem 9.2.4. Let f and g be modulus functions that satisfy the $\triangle_2$-condition. If X is any of the spaces ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0, m^I_{2\mathcal{Z}}$ and $m^I_{2\mathcal{Z}_0}$, then the following assertions hold
(a)$X(g) \subseteq X(f.g)$,
(b)$X(f) \cap X(g) \subseteq X(f+g)$

Proof. (a) Let $(x_{ij}) \in {}_2\mathcal{Z}^I_0(g)$. Then

$$I - \lim_{ij} g(|x'_{ij}|) = 0 \qquad [9.5]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$. Write $y_{ij} = g(|x'_{ij}|)$ and consider

$$\lim_{i,j} f(y_{ij}) = \lim_{i,j} f(y_k)_{y_{ij}<\delta} + \lim_{i,j} f(y_{ij})_{y_{ij}>\delta}$$

We have

$$\lim_{i,j} f(y_{ij}) \leq f(2) \lim_{i,j}(y_{ij}) \qquad [9.6]$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Since f is non-decreasing, it follows that

$$f(y_{ij}) < f(1 + \frac{y_{ij}}{\delta}) < \frac{1}{2}f(2) + \frac{1}{2}f(\frac{2y_k}{\delta})$$

Since f satisfies the $\triangle_2$-condition, we have

$$f(y_{ij}) < \frac{1}{2}K\frac{y_{ij}}{\delta}f(2) + \frac{1}{2}K\frac{y_{ij}}{\delta}f(2) = K\frac{y_{ij}}{\delta}f(2)$$

Hence

$$\lim_{i,j} f(y_{ij}) \leq max(1, K)\delta^{-1}f(2)\lim_{i,j}(y_{ij}). \qquad [9.7]$$

From [9.5], [9.6] and [9.7], we have $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f.g)$. Thus ${}_2\mathcal{Z}_0^I(g) \subseteq {}_2\mathcal{Z}_0^I(f.g)$. The other cases can be established following similar technique.

(b) Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f) \cap {}_2\mathcal{Z}_0^I(g)$. Then $I - \lim_{i,j} f(|x'_{ij}|) = 0$ and $I - \lim_{i,j} g(|x'_{ij}|) = 0$

The rest of the proof follows from the following equality

$$\lim_{i,j}(f + g)(|x'_{ij}|) = \lim_{ij} f(|x'_{ij}|) + \lim_{i,j} g(|x'_{ij}|).$$

Corollary 9.2.5. $X \subseteq X(f)$ for X = ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}_0^I, m^I_{{}_2\mathcal{Z}}$ and $m^I_{{}_2\mathcal{Z}_0}$.

Theorem 9.2.6. The spaces ${}_2\mathcal{Z}_0^I(f)$ and $m^I_{{}_2\mathcal{Z}_0}(f)$ are solid and monotone.

Proof. We shall prove the result for the sequence space ${}_2\mathcal{Z}_0^I(f)$. Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f)$. Then

$$I - \lim_{i,j} f(|x'_{ij}|) = 0. \qquad [9.8]$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [9.8] and the following inequality

$$f(|\alpha_{ij}x_{ij}^{'}|) \leq |\alpha_{ij}|f(|x_{ij}^{'}|) \leq f(|x_{ij}^{'}|)\ for\ all\ (i,j) \in \mathbb{N} \times \mathbb{N}.$$

That the space ${}_2\mathcal{Z}_0^I(f)$ is monotone follows from the Lemma 1.12. For $m^I_{{}_2\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 9.2.7. The spaces ${}_2\mathcal{Z}^I(f)$ and $m^I_{{}_2\mathcal{Z}}(f)$ are neither solid nor monotone in general .

Proof. We prove this result by providing a counter example. Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows

Let $(x_{ij}) \in X$ and let $(y_{ij}) \in X_K$ be such that

$$(y_{ij}) = \begin{cases} (x_{ij}) & \text{if i+j is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$ but its K-stepspace preimage does not belong to ${}_2\mathcal{Z}^I(f)$. Thus ${}_2\mathcal{Z}^I(f)$ is not monotone. Hence ${}_2\mathcal{Z}^I(f)$ is not solid.

Theorem 9.2.8. The spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are sequence algebras.

Proof. We prove that the sequence space ${}_2\mathcal{Z}_0^I(f)$ is a sequence algebra. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I(f)$. Then

$$I - \lim f(|x_{ij}^{'}|) = 0 \quad and \quad I - \lim f(|y_{ij}^{'}|) = 0$$

Then we have

$$I - \lim f(|x_{ij}^{'}.y_{ij}^{'}|) = 0$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I(f)$ is a sequence algebra. For the space ${}_2\mathcal{Z}_0^I(f)$, the result can be proved similarly.

Theorem 9.2.9. The spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not convergence free in general.

Proof. We give a counter example to prove this result.

Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i+j} \quad and \ \ y_{ij} = i + j \ \text{ for all } (i,j) \in \mathbb{N} \times \mathbb{N}.$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$. Hence the spaces ${}_2\mathcal{Z}_0^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not convergence free.

Theorem 9.2.10. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f(x) = x$ for all $x \in [0, \infty)$. If

$$x_{ij} = \begin{cases} 1, & \text{for } (i,j) \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.14 $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(m)\pi(n)} \notin {}_2\mathcal{Z}^I(f)$ and $x_{\pi(m)\pi(n)} \notin {}_2\mathcal{Z}_0^I(f)$. Hence ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not symmetric.

Theorem 9.2.11. Let f be a modulus function. Then ${}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f) \subset {}_2\mathcal{Z}_\infty^I(f)$.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f(|x'_{ij} - L|) = 0$$

We have $f(|x'_{ij}|) \leq f(|x'_{ij} - L|) + f(|L|)$. Taking the supremum over (i,j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I(f)$. The inclusion ${}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f)$ is obvious.

Theorem 9.2.12. The function $\hbar : m^I_{2\mathcal{Z}}(f) \to \mathbb{R}$ is the Lipschitz function, where $m^I_{2\mathcal{Z}}(f) = {}_2\mathcal{Z}_\infty^I(f) \cap {}_2\mathcal{Z}^I(f)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I{}_{2\mathcal{Z}}(f)$, $x \neq y$. Then the sets

$$A_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < ||x - y||_*\} \in m^I_{2\mathcal{Z}}(f),$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_k - \hbar(y)| < ||x - y||_*\} \in m^I_{2\mathcal{Z}}(f).$$

Hence also $B = B_x \cap B_y \in m^I_{2\mathcal{Z}}(f)$, so that $B \neq \Phi$. A Now taking (i,j) in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y_{ij} - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{2\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 9.2.13. If $x, y \in m^I_{2\mathcal{Z}}(f)$, then $(x.y) \in m^I_{2\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \epsilon\} \in m^I_{2\mathcal{Z}}(f),$$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

Now,

$$\begin{aligned} |x_{ij}y_{ij} - \hbar(x)\hbar(y)| &= |x_{ij}y_{ij} - x_{ij}\hbar(y) + x_{ij}\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_{ij}||y_{ij} - \hbar(y)| + |\hbar(y)||x_{ij} - \hbar(x)| \qquad [9.9] \end{aligned}$$

As $m^I_{2\mathcal{Z}}(f) \subseteq {}_2\mathcal{Z}^I_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|x_{ij}| < M$ and $|\hbar(y)| < M$.

Using eqn[9.9] we get

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon \;=\; 2M\epsilon$$

for all $(i,j) \in B_x \cap B_y \in m^I(f)$. Hence $(x.y) \in m^I_{2\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{2\mathcal{Z}_0}(f)$ the result can be proved similarly.

Chapter 10

Zweier I-Convergent Double Sequence Spaces Defined by Orlicz Function

10.1 Introduction

Recently Vakeel. A. Khan et. al.[37] introduced and studied the following classes of sequence spaces:

$$\mathcal{Z}^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^{/} - L|}{\rho}) = 0 \text{ for some L and } \rho > 0\},$$

$$\mathcal{Z}_0^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^{/}|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

$$\mathcal{Z}_\infty^I(M) = \{(x_k) \in \omega : \sup_k M(\frac{|x_k^{/}|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

Also we denote by

$$m_{\mathcal{Z}}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}^I(M)$$

and

$$m_{\mathcal{Z}_0}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}_0^I(M).$$

10.2 Main Results

In this Chapter we introduce the following classes of Zweier I-Convergent double sequence spaces defined by the Orlicz function.

$${}_2\mathcal{Z}^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|x_{ij}^{'} - L|}{\rho}) = 0$$

$$\text{for some L} \in \mathbb{C} \text{ , and } \rho > 0\},$$

$${}_2\mathcal{Z}_0^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|x_{ij}^{'}|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

"Mazur and Orlicz are direct pupils of Banch; they represent the theory of operations today in poland and their names cover of "Studia Mathematica" indicate direct continuation of Banach's scientific programme."-Hugo Steinhauss

$$
{}_2\mathcal{Z}^I_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} : \text{there exist } K > 0 :
$$
$$
M(\frac{|x'_{ij}|}{\rho}) \geq K \text{ for some } \rho > 0 \in I\}.
$$
$$
{}_2\mathcal{Z}_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} M(\frac{|x'_{ij}|}{\rho}) < \infty\}
$$

Also we denote by

$$
m^I_{2\mathcal{Z}}(M) = {}_2\mathcal{Z}^I_\infty(M) \cap {}_2\mathcal{Z}^I(M)
$$

and

$$
m^I_{2\mathcal{Z}_0}(M) = {}_2\mathcal{Z}^I_\infty(M) \cap {}_2\mathcal{Z}^I_0(M).
$$

Throughout the chapter, for the sake of convenience, we will denote by $Z^p(x_k) = x'$, $Z^p(y_k) = y'$, $Z^p(z_k) = z'$ for $x, y, z \in \omega$.

Theorem 10.2.1. For any Orlicz function M, the classes of sequences ${}_2\mathcal{Z}^I(M)$, ${}_2\mathcal{Z}^I_0(M)$, ${}_2m^I_{\mathcal{Z}}(M)$ and ${}_2m^I_{\mathcal{Z}_0}(M)$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(M)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I(M)$ and let α, β be scalars. Then there exists positive numbers ρ_1 and ρ_2 such that

$$
I - \lim M(\frac{|x'_{ij} - L_1|}{\rho_1}) = 0, \text{ for some } L_1 \in \mathbb{C} \ ;
$$
$$
I - \lim M(\frac{|y'_{ij} - L_2|}{\rho_2}) = 0, \text{ for some } L_2 \in \mathbb{C} \ ;
$$

That is for a given $\epsilon > 0$, we have

$$
A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|x'_{ij} - L_1|}{\rho_1}) > \frac{\epsilon}{2}\} \in I, \qquad [10.1]
$$
$$
A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|y'_{ij} - L_2|}{\rho_2}) > \frac{\epsilon}{2}\} \in I. \qquad [10.2]
$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex function, we have

$$\begin{aligned} & M(\frac{|(\alpha x_{ij}^{'} + \beta y_{ij}^{'}) - (\alpha L_1 + \beta L_2)|}{\rho_3}) \\ \leq & M(\frac{|\alpha||x_{ij}^{'} - L_1|}{\rho_3}) + M(\frac{|\beta||y_{ij}^{'} - L_2|}{\rho_3}). \\ \leq & M(\frac{|x_{ij}^{'} - L_1|}{\rho_1}) + M(\frac{|y_{ij}^{'} - L_2|}{\rho_2}) \end{aligned}$$

Now, by [10.1] and [10.2],

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|(\alpha x_{ij}^{'} + \beta y_{ij}^{'}) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in_2 \mathcal{Z}^I(M)$. Hence ${}_2\mathcal{Z}^I(M)$ is a linear space.

Theorem 10.2.2. The spaces ${}_2m^I_{\mathcal{Z}}(M)$ and ${}_2m^I_{\mathcal{Z}_0}(M)$ are Banach spaces normed by

$$||x_{ij}|| = \inf\{\rho > 0 : \sup_{i,j} M(\frac{|x_{ij}|}{\rho}) \leq 1\}$$

Proof. Proof of this result is easy in view of the existing techniques and therefore is omitted.

Theorem 10.2.3. Let M_1 and M_2 be Orlicz functions that satisfy the $\triangle_2$-condition. Then
(a) $X(M_2) \subseteq X(M_1.M_2)$;
(b) $X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ For $X = {}_2\mathcal{Z}^I, {}_2\mathcal{Z}^I_0, {}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$.

Proof. (a) Let $(x_{ij}) \in {}_2\mathcal{Z}^I_0(M_2)$. Then there exists $\rho > 0$ such that

$$I - \lim_{i,j} M_2(\frac{|x_{ij}^{'}|}{\rho}) = 0 \qquad [10.3]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{ij} = M_2(\frac{|x'_{ij}|}{\rho})$ and consider for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ we have

$$\lim_{0 \leq y_{ij} \leq \delta} M_1(y_{ij}) = \lim_{y_{ij} \leq \delta} M_1(y_{ij}) + \lim_{y_{ij} > \delta} M_1(y_{ij}).$$

We have

$$\lim_{y_{ij} \leq \delta} M_1(y_{ij}) \leq M_1(2) \lim_{y_{ij} \leq \delta} (y_{ij}). \qquad [10.4]$$

For $(y_{ij}) > \delta$, we have

$$(y_{ij}) < (\frac{y_{ij}}{\delta}) < 1 + (\frac{y_{ij}}{\delta}).$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(y_{ij}) < M_1(1 + (\frac{y_{ij}}{\delta})) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_{ij}}{\delta})$$

Since M_1 satisfies the $\triangle_2$-condition, we have

$$M_1(y_{ij}) < \frac{1}{2} K(\frac{y_{ij}}{\delta}) M_1(2) + \frac{1}{2} K(\frac{y_{ij}}{\delta}) M_1(2) = K(\frac{y_{ij}}{\delta}) M_1(2).$$

Hence

$$\lim_{y_{ij} > \delta} M_1(y_{ij}) \leq max(1, K\delta^{-1} M_1(2)) \lim_{y_{ij} > \delta} (y_{ij}). \qquad [10.5]$$

From [10.3], [10.4] and [10.5], we have $(x_{ij}) \in \mathcal{Z}_0^I(M_1).(M_2)$. Thus

$$\mathcal{Z}_0^I(M_2) \subseteq \mathcal{Z}_0^I(M_1.M_2).$$

The other cases can be proved similarly.

(b) Let $(x_k) \in \mathcal{Z}_0^I(M_1) \cap \mathcal{Z}_0^I(M_2)$. Then there exists $\rho > 0$ such that $I - \lim_k M_1(\frac{|x_k^/|}{\rho}) = 0$ and $I - \lim_k M_2(\frac{|x_k^/|}{\rho}) = 0$. The rest of the proof follows from the following equality

$$\lim_{k\in\mathbb{N}}(M_1 + M_2)(\frac{|x_k^/|}{\rho}) = \lim_{k\in\mathbb{N}} M_1(\frac{|x_k^/|}{\rho}) + \lim_{k\in\mathbb{N}} M_2(\frac{|x_k^/|}{\rho})$$

Theorem 10.2.4. The spaces ${}_2\mathcal{Z}_0^I(M)$ and ${}_2m_{\mathcal{Z}_0}^I(M)$ are solid and monotone .

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I(M)$. For $m_{\mathcal{Z}_0}^I(M)$ the result can be proved similarly. Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(M)$. Then there exists $\rho > 0$ such that

$$I - \lim_{i,j} M(\frac{|x_{ij}^{'}|}{\rho}) = 0 \qquad [10.6]$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [10.6] and the following inequality for all

$$M(\frac{|\alpha_{ij}x_{ij}^{'}|}{\rho}) \leq |\alpha_{ij}|M(\frac{|x_{ij}^{'}|}{\rho}) \leq M(\frac{|x_{ij}^{'}|}{\rho}).$$

By Lemma 1.12, a sequence space E is solid implies that E is monotone. We have the space ${}_2\mathcal{Z}_0^I(M)$ is monotone.

Theorem 10.2.5. The spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2m_{\mathcal{Z}}^I(M)$ are neither solid nor monotone in general.

Proof. Here we give a counter example. Let $I = I_\delta$ and $M(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(M)$ of $X(M)$ defined as follows, Let $(x_{ij}) \in X(M)$ and let $(y_{ij}) \in X_K(M)$ be such that

$$y_{ij} = \begin{cases} x_{ij}, & \text{if (i+j) is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence x_{ij} defined by $x_{ij} = 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$ but its K-stepspace preimage does not belong to ${}_2\mathcal{Z}^I(M)$. Thus ${}_2\mathcal{Z}^I(M)$ is not monotone.

Hence ${}_2\mathcal{Z}^I(M)$ is not solid.

Theorem 10.2.6. The spaces ${}_2\mathcal{Z}_0^I(M)$ and ${}_2\mathcal{Z}^I(M)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $M(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i+j} \quad and \;\; y_{ij} = i + j$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}_0^I(M)$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}_0^I(M)$. Hence the spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}_0^I(M)$ are not convergence free.

Theorem 10.2.7. The spaces ${}_2\mathcal{Z}_0^I(M)$ and ${}_2\mathcal{Z}^I(M)$ are sequence algebras.

Proof. We prove that ${}_2\mathcal{Z}_0^I(M)$ is a sequence algebra. For the space ${}_2\mathcal{Z}^I(M)$, the result can be proved similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I(M)$. Then

$$I - \lim M(\frac{|x'_{ij}|}{\rho_1}) = 0$$

and

$$I - \lim M(\frac{|y'_{ij}|}{\rho_2}) = 0$$

Let $\rho = \rho_1.\rho_2 > 0$. Then we can show that

$$I - \lim M(\frac{|(x'_{ij}.y'_{ij})|}{\rho}) = 0.$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I(M)$. Hence ${}_2\mathcal{Z}_0^I(M)$ is a sequence algebra.

Theorem 10.2.8. Let M be an Orlicz function. Then the inclusions

$$ {}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M) \subset {}_2\mathcal{Z}_\infty^I(M) $$

hold.

Proof: Let $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$ I - \lim M(\frac{|x'_{ij} - L|}{\rho}) = 0. $$

We have $M(\frac{|x'_{ij}|}{2\rho}) \leq \frac{1}{2}M(\frac{|x'_{ij}-L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho})$. Taking supremum over (i,j) both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I(M)$. The inclusion ${}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M)$ is obvious.

Theorem 10.2.9. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}_0^I(M)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $M(x) = x$ for all $x = (x_{ij})$. If

$$ x_{ij} = \begin{cases} 1, & \text{for } i, j \in A, \\ 0, & otherwise. \end{cases} $$

Then

$$ (x_{ij}) \in {}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M), $$

by lemma 1.14. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$.

Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$ \pi(k) = \begin{cases} \phi(k), & \text{for} k \in K, \\ \psi(k), & otherwise. \end{cases} $$

is a permutation on $\mathbb{N}$, but $(x_{\pi(i)\pi(j)}) \notin {}_2\mathcal{Z}^I(M)$ and $(x_{\pi(i)\pi(j)}) \notin {}_2\mathcal{Z}_0^I(M)$. Hence ${}_2\mathcal{Z}_0^I(M)$ and ${}_2\mathcal{Z}^I(M)$ are not symmetric.

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Introduction

The concept of ideal convergence is a generalization of statistical convergence, and any concept involving ideal convergence plays a vital role not only in the pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling and motion planning in robotics.

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